What is the Limit of a Sequence of Graphs?

Vilas Winstein

June 22, 2021

Why is...?

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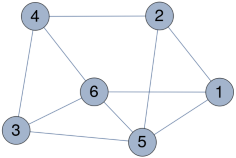

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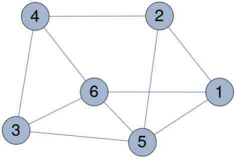

$$\longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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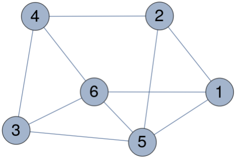

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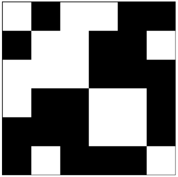

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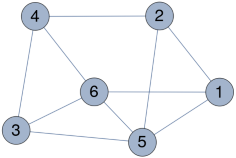

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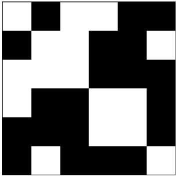

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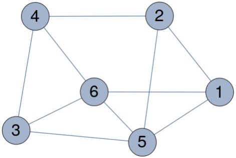

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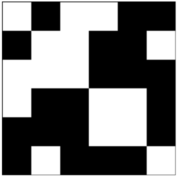

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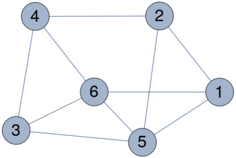

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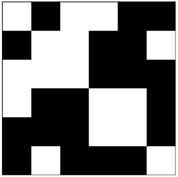

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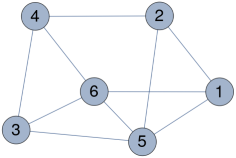

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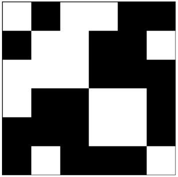

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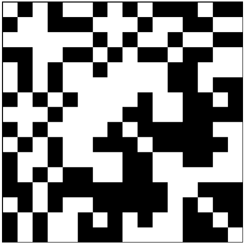

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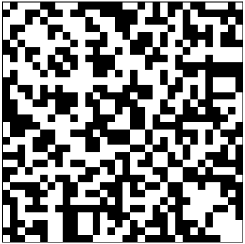

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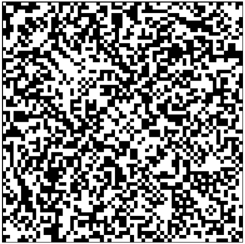

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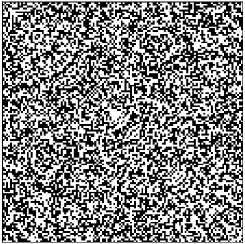

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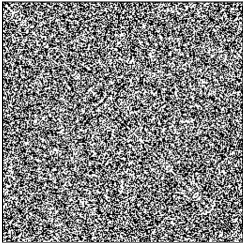

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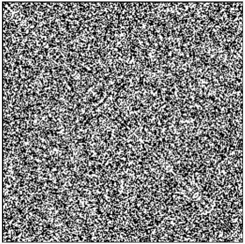

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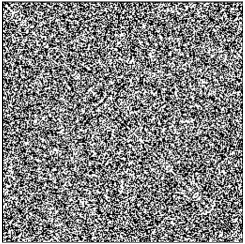

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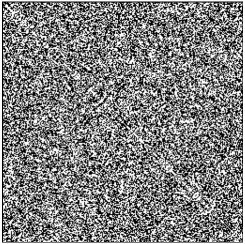

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The limit object is a function $[0,1]^2 \to [0,1]$ which does not come from any finite graph, since it takes values outside of $\{0,1\}$.

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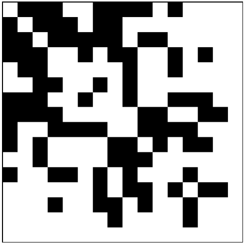

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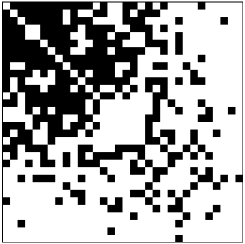

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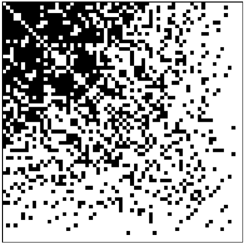

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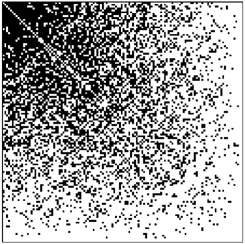

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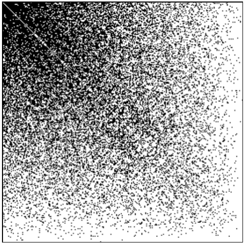

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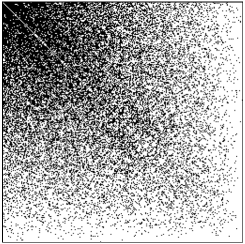

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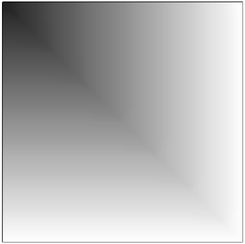

$f(x,y) = 1 - \max(x,y)$

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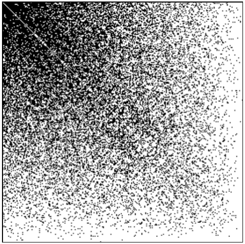

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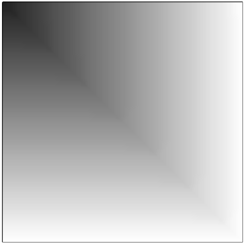

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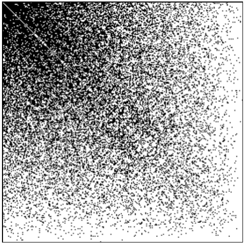

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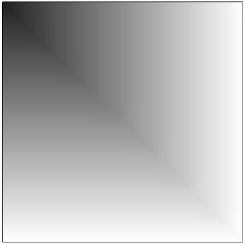

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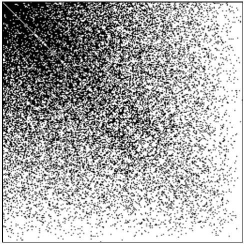

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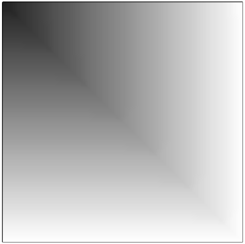

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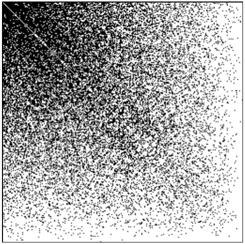

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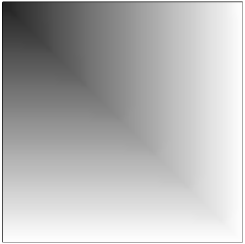

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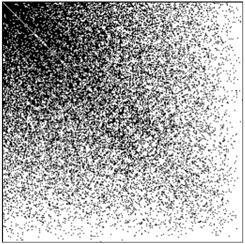

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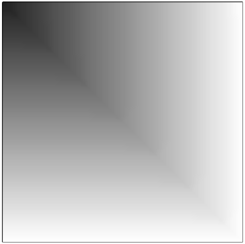

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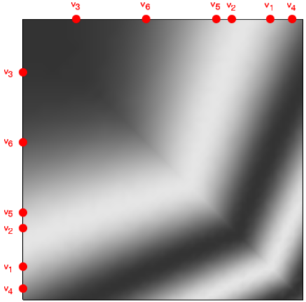

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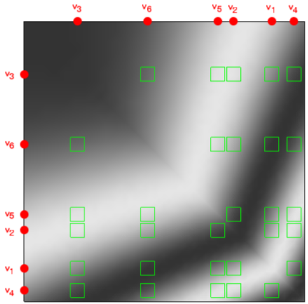

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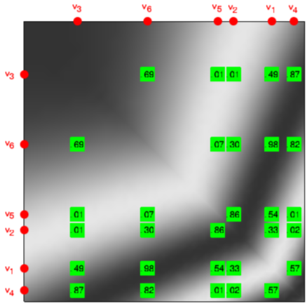

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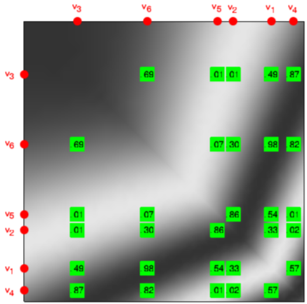

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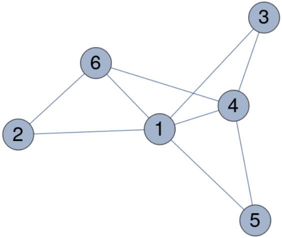

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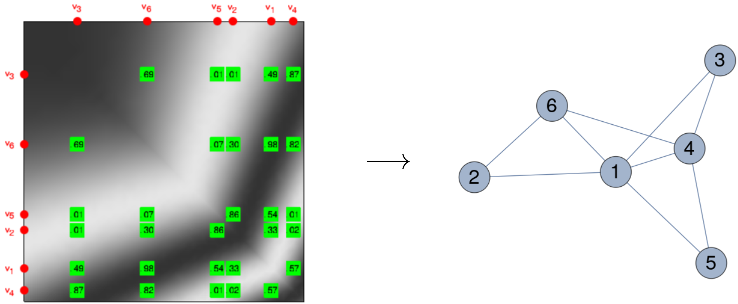

As $n \to \infty$, the sequence of sampled graphs will (almost surely) converge (in the cut norm topology) to the graphon we started with.

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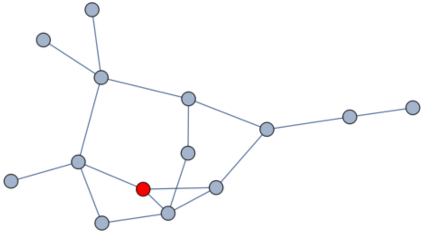

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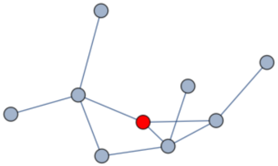

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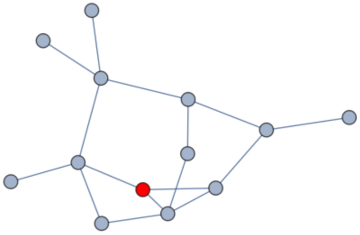

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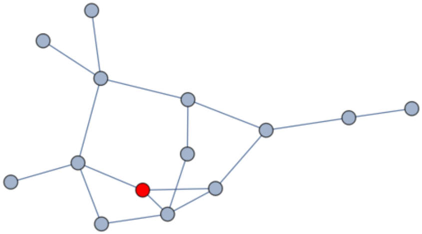

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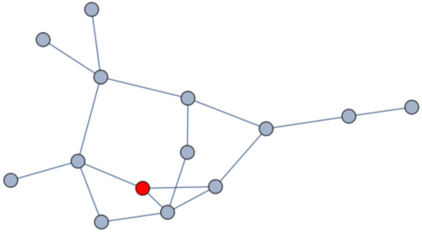

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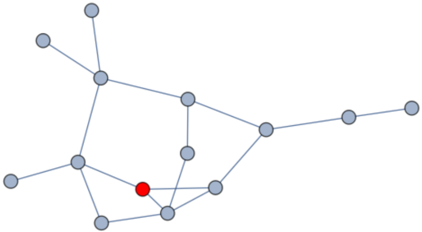

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Note that we still keep track of the original root in the r-balls.

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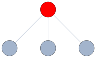
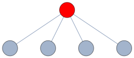
$d_{\text{local}} \Bigg($  ,  $\Bigg) = \dfrac{1}{2},$

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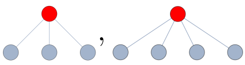
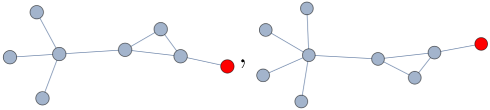
$d_{\text{local}}\Big($  $\Big) = \frac{1}{2}$, and

$d_{\text{local}}\Big($  $\Big) = \frac{1}{16}$.

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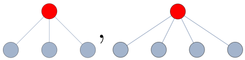
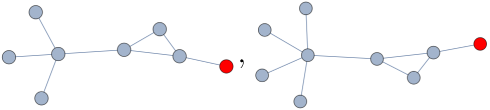
$d_{\text{local}}\Big($  $\Big) = \frac{1}{2}$, and

$d_{\text{local}}\Big($  $\Big) = \frac{1}{16}$.

By the way, if (G, ρ) and (G', ρ') are equal (isomorphic as *rooted* graphs), then their distance is 0.

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For totally boundedness, note that there are only finitely many different graphs $(G_1^r, \rho_1^r), \ldots, (G_{k_r}^r, \rho_{k_r}^r)$ with radius $\leq r$ (and degree bound D).

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For completeness, note that if a sequence (G_n, ρ_n) is Cauchy, then for every radius r, the r-balls $B_r(G_n, \rho_n)$ are eventually constant. So we can define a limiting graph whose r-ball is this constant graph, for every r (note that $\mathcal{G}_D^\bullet$ does not exclude infinite graphs).

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$$C_i^r = \{(G, \rho) \in \mathcal{G}_D^\bullet : B_r(G, \rho) = (G_i^r, \rho_i^r)\}$$

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Examples of rooted graph limits

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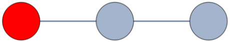

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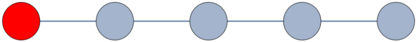

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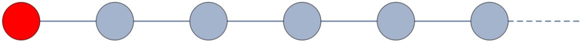

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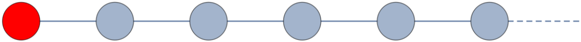

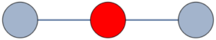

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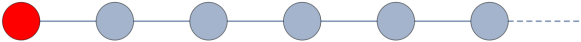

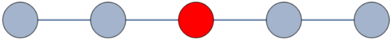

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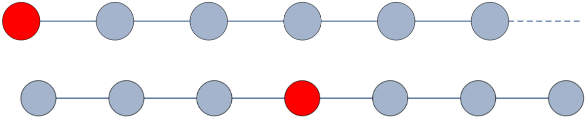

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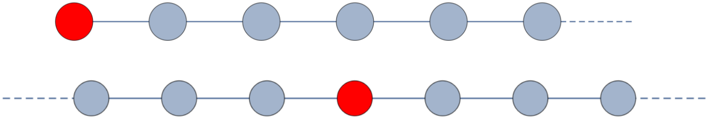

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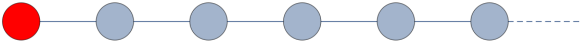

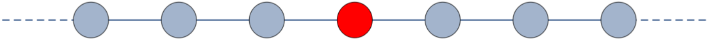

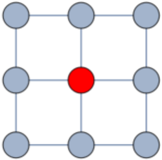

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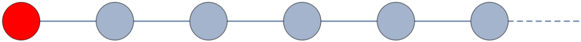

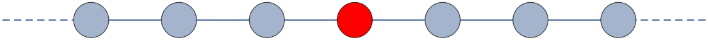

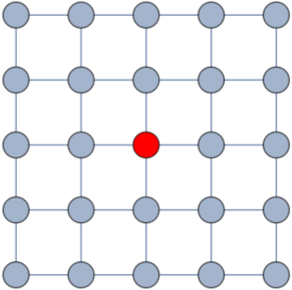

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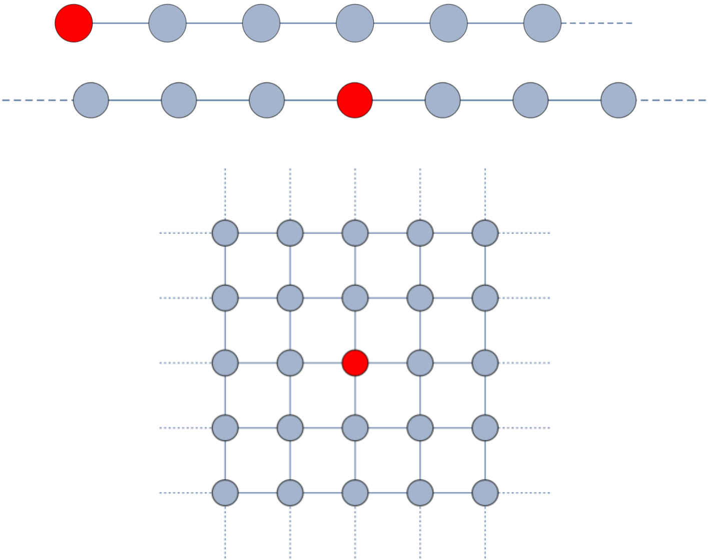

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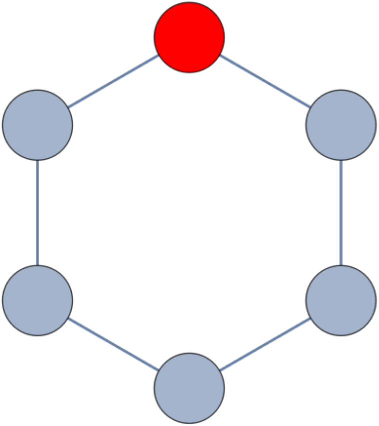

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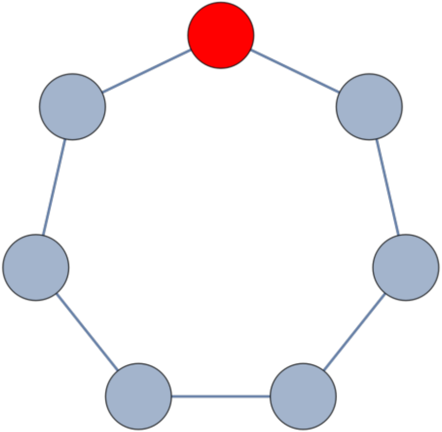

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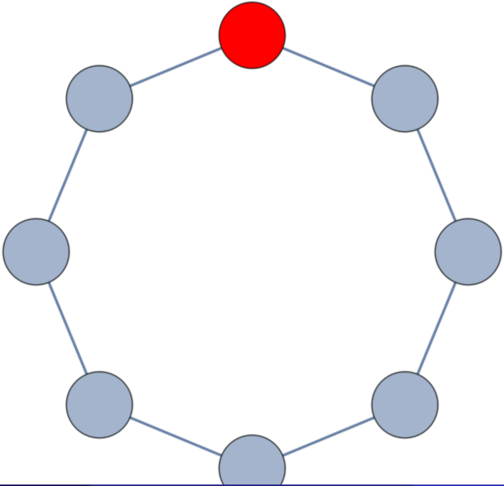

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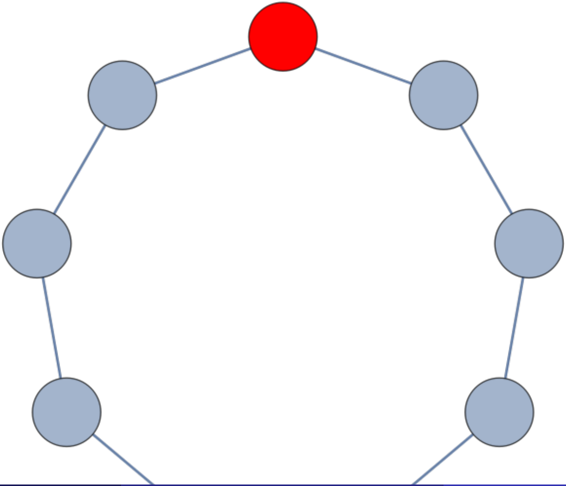

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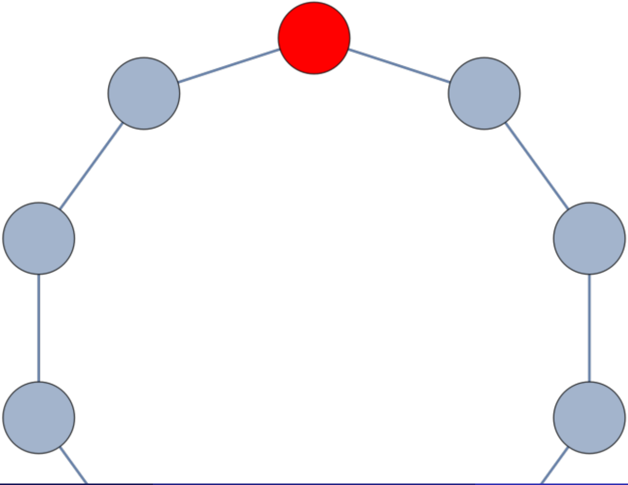

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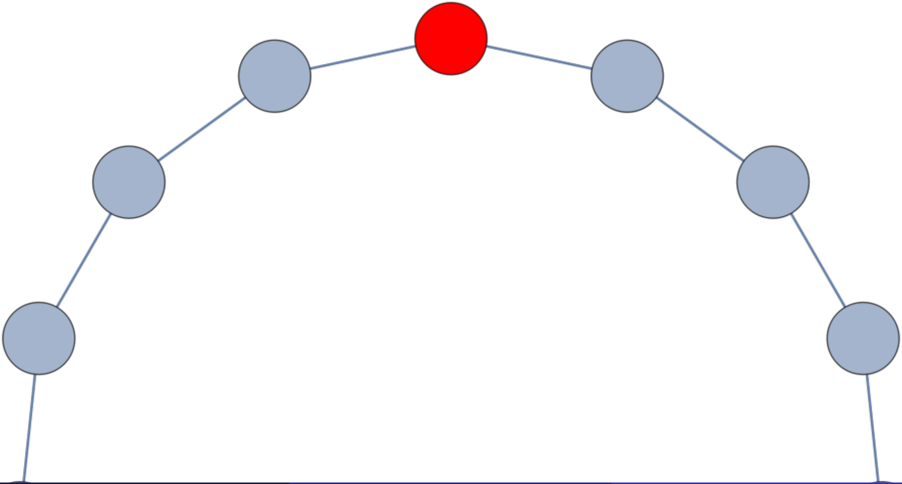

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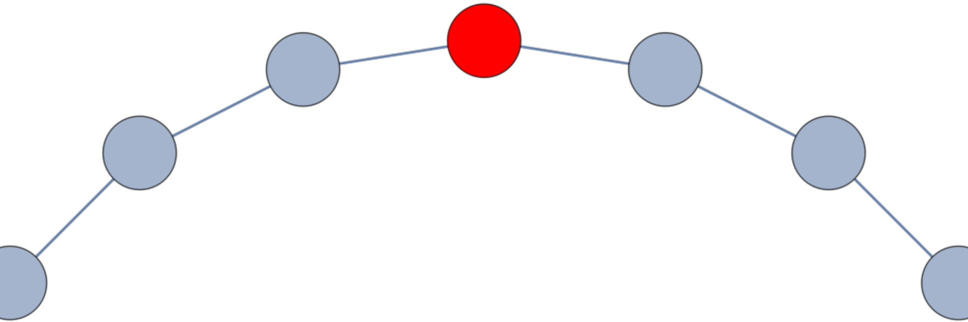

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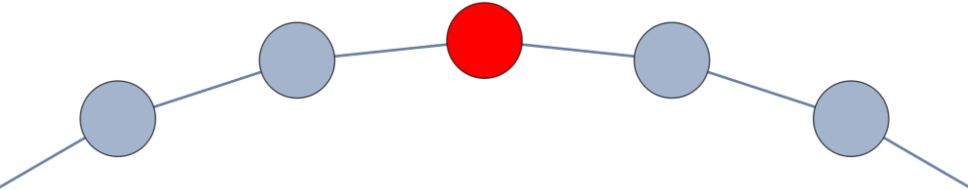

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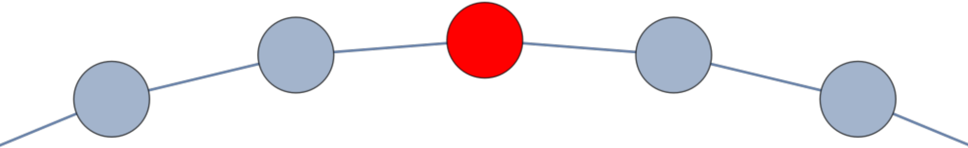

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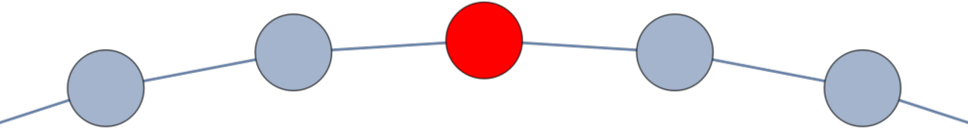
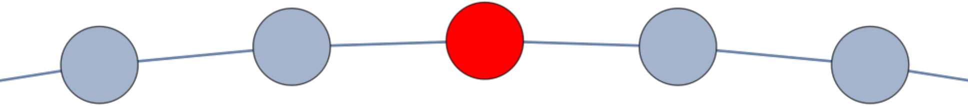


Examples of rooted graph limits

Bi-infinite path (again)



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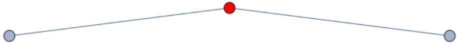

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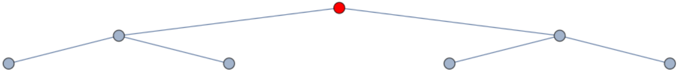

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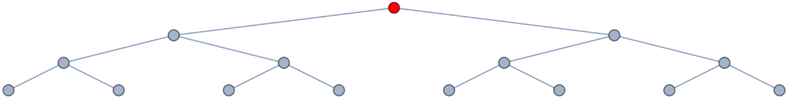

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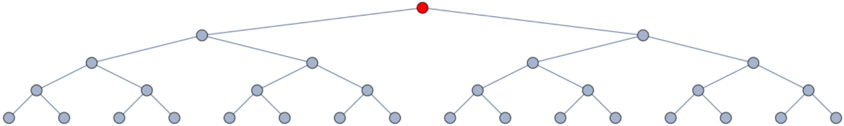

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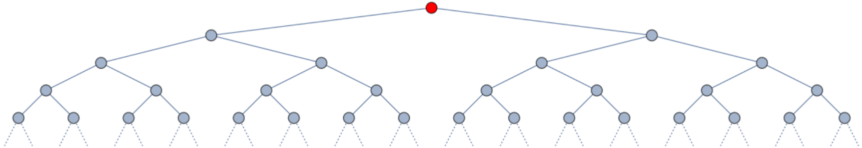

Infinite Binary Tree

Examples of rooted graph limits

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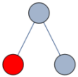

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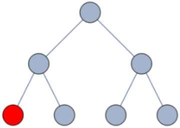

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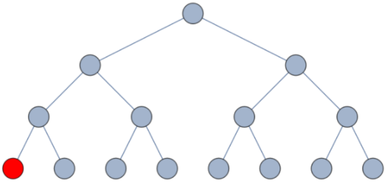

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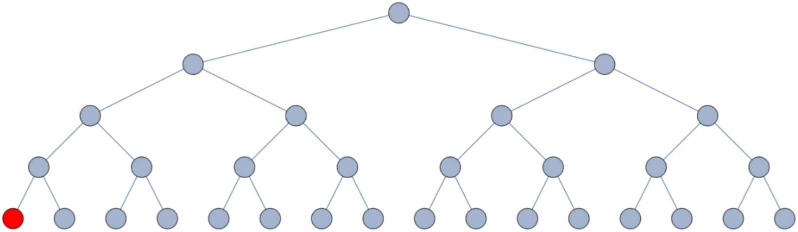

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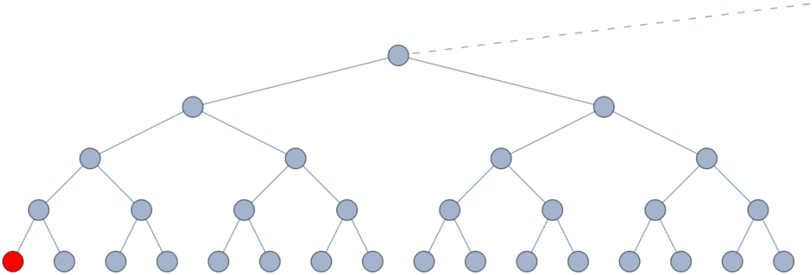

Sierpiński Tree

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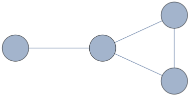

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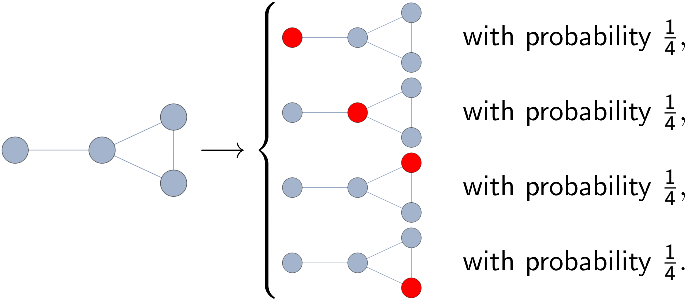

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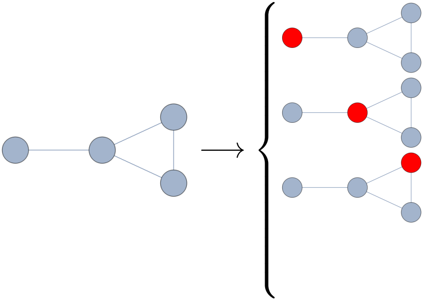

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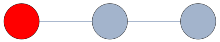

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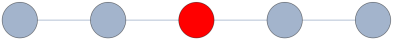

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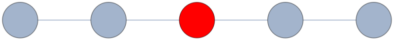

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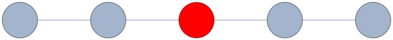

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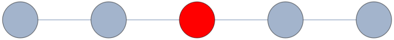

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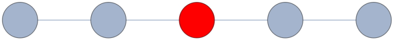

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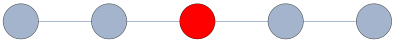

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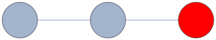

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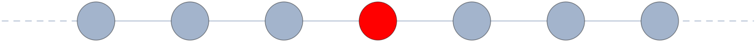

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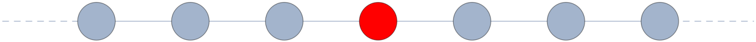

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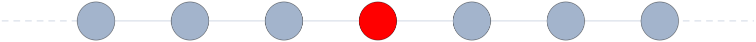

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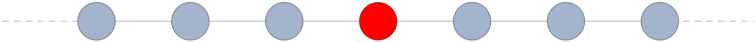

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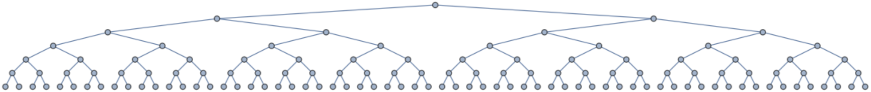

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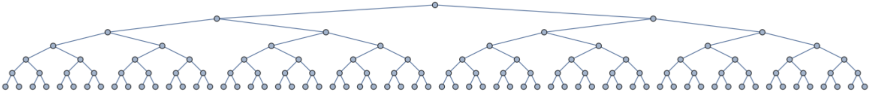

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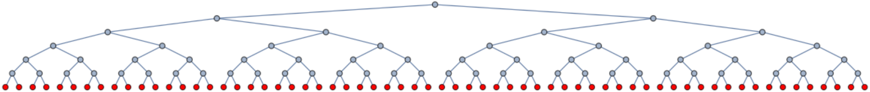

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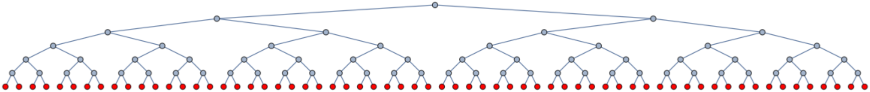

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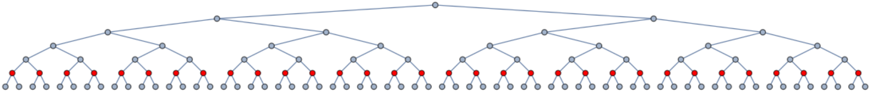

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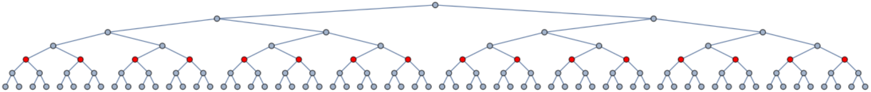

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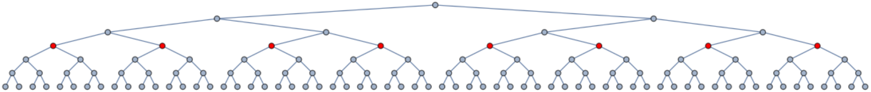

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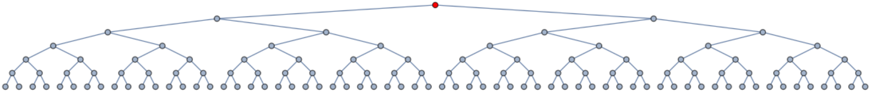

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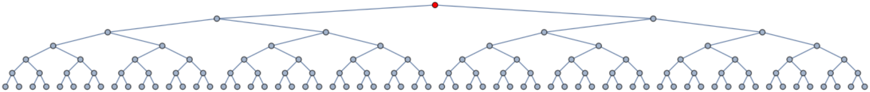

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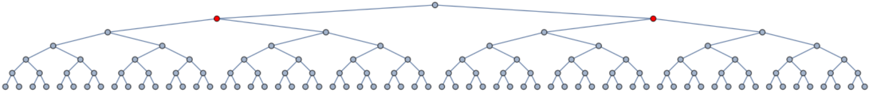

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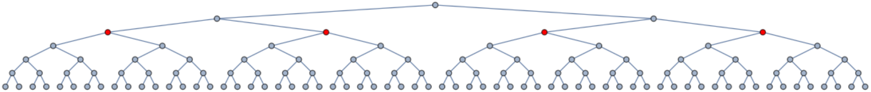

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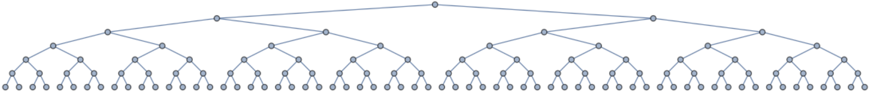

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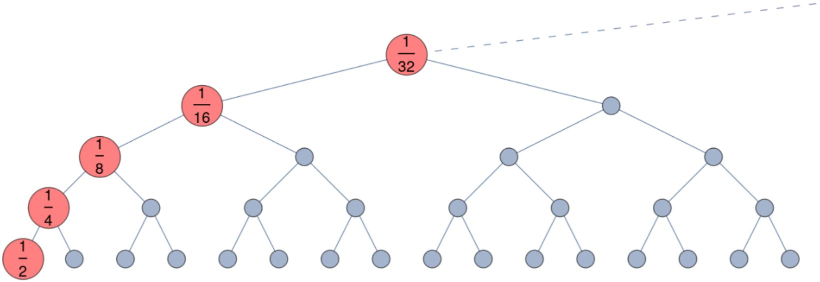

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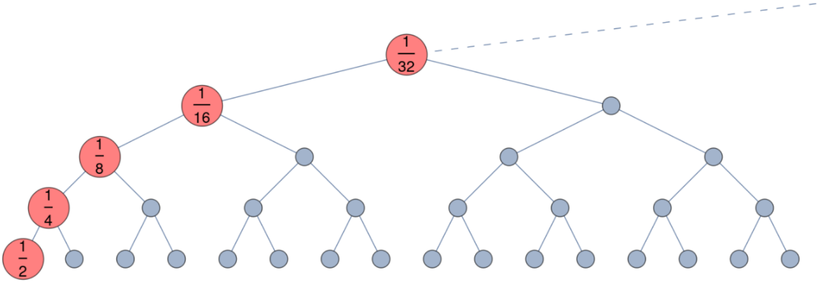

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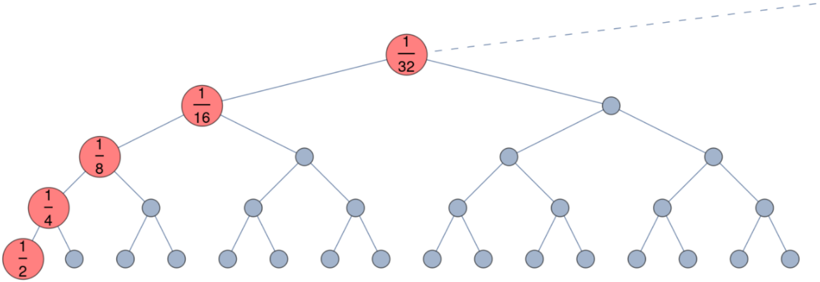

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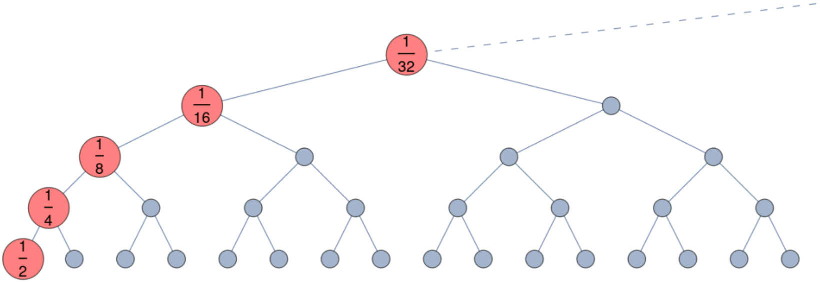

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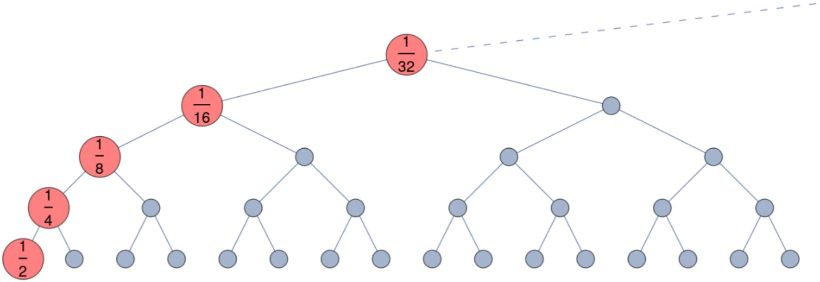

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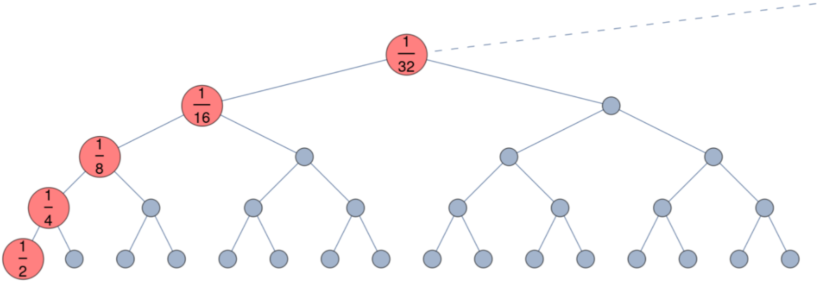

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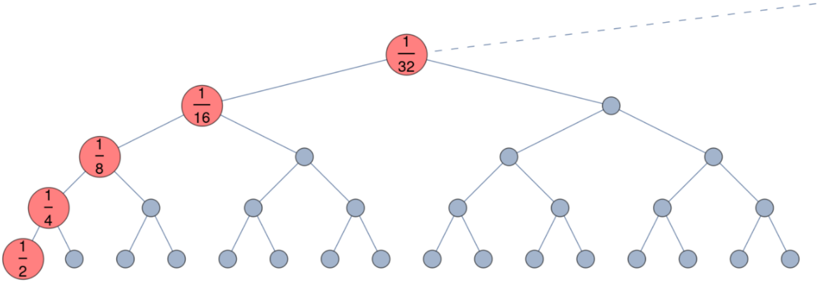

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The left-hand side is the expected amount of "mass" sent by the (random) root ρ. And the right-hand side is the expected amount received by ρ.

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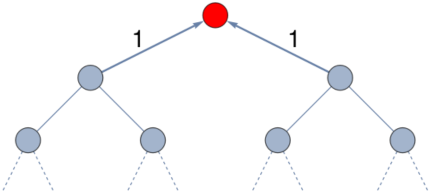

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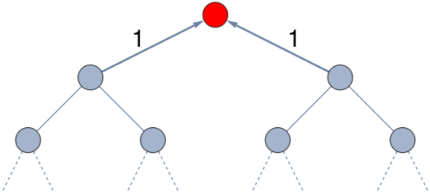

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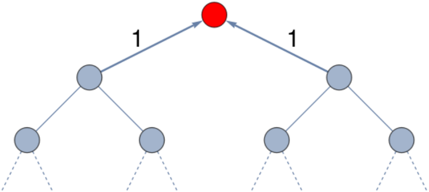

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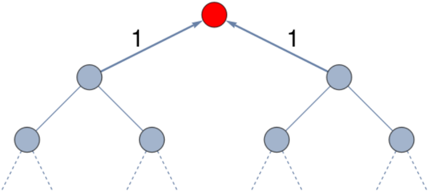

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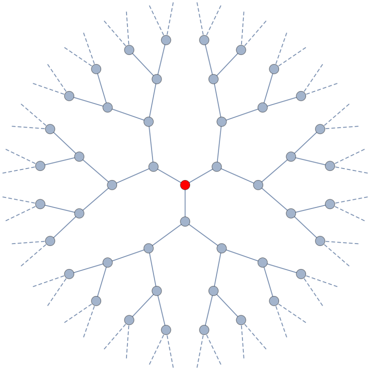

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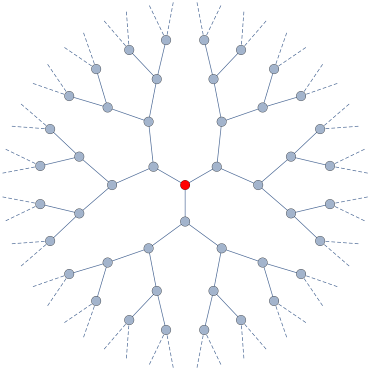

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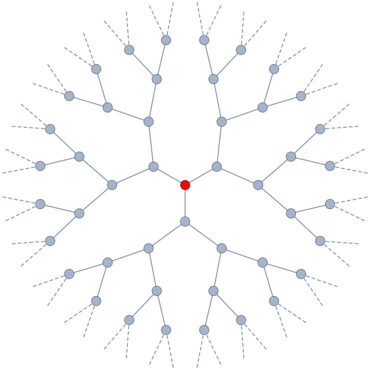

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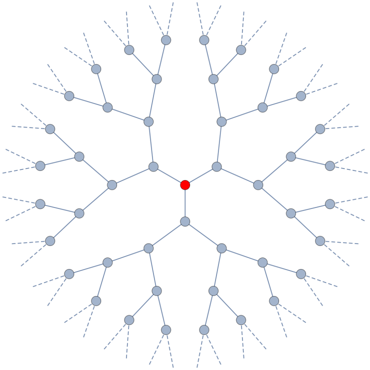

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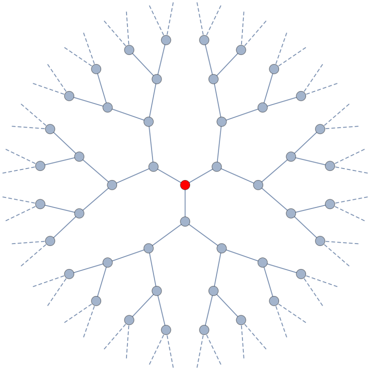

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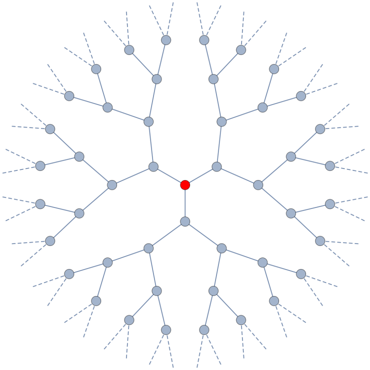

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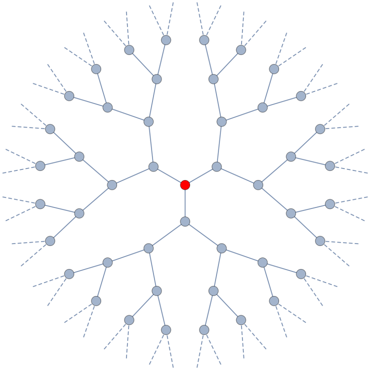

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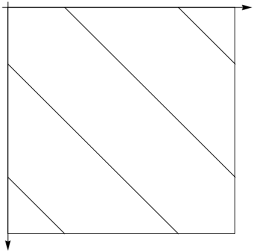

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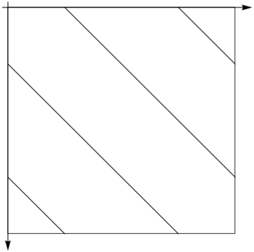

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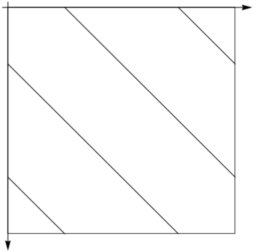

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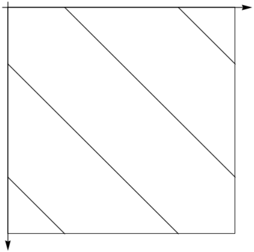

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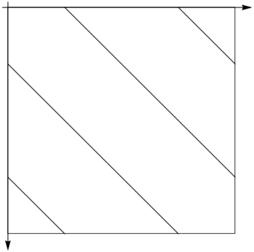

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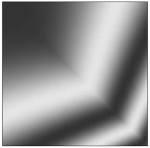

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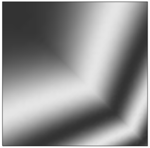

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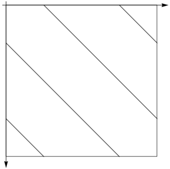

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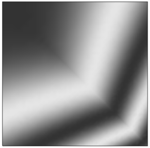
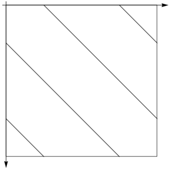
Both objects can be thought of as measures on $[0,1]^2$.



Graphons are absolutely continuous.



Graphings are highly singular.

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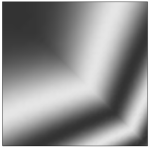

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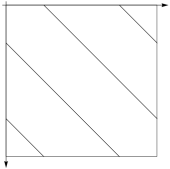

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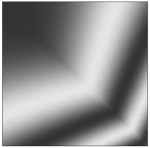

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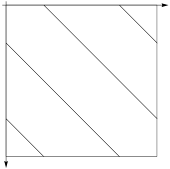

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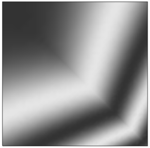

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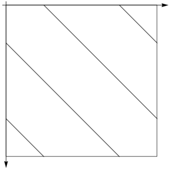

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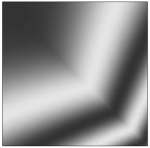

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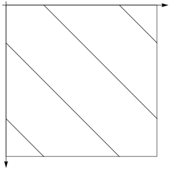

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References

[1] Aldous, David, and Russell Lyons. "Processes on unimodular random networks." Electronic Journal of Probability 12 (2007): 1454-1508.

[2] Benjamini, Itai, and Oded Schramm. "Recurrence of distributional limits of finite planar graphs." Selected Works of Oded Schramm. Springer, New York, NY, 2011. 533-545.

[3] Curien, Nicolas. Random Graphs: the Local Convergence Point of View. Lecture Notes. September 22, 2017.

[4] Lovász, László. Continuous limits of finite structures. The Abel Lectures. The Abel Prize YouTube Channel. May 26, 2021.

[5] Lovász, László. Large networks and graph limits. Vol. 60. American Mathematical Soc., 2012.