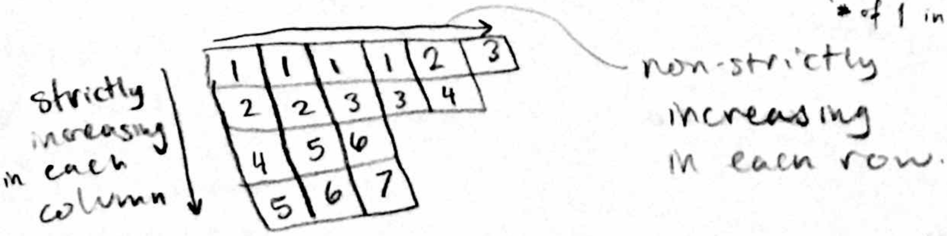
# Semistandard Young Tableaux

- Young diagram filled with integers. has Shape & Weight (or type), both partitions of $n$.

  eg. a SSYT of shape $(6,5,3,3)$ and weight $(4,3,3,2,2,2,1)$:
  
  \# of 1 in tableau, etc.




- $K_{\mu\lambda}$ = the number of SSYT of weight $\lambda$ and shape $\mu$.

  eg $K_{(2,2,2)(3,2,1)} = 2$

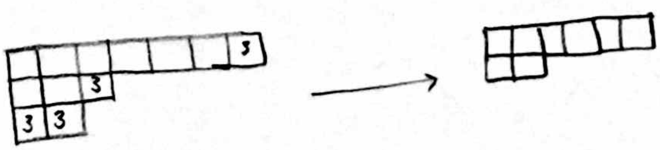
Dominance Order: $\lambda, \mu \vdash n$. $\mu \geq \lambda$ if $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i \quad \forall i$.

**Lemma:** $K_{\lambda\lambda} = 1$, and $K_{\mu\lambda} > 0$ iff $\mu \geq \lambda$.

Proof: ($\Rightarrow$) all of the $1^s, \ldots, i^s$ must be in the first $i$ rows so $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$.

($\Leftarrow$): Example as a proof (this can be generalized). Work by induction on $n$.

$\lambda = (4,4,4)$, $\mu = (7,3,2)$.



$\lambda' = (4,4)$, $\mu' = (6,2)$. Induction says we can fill in the resulting table.

**Theorem** (Robinson – Schensted – Knuth Correspondence): Let $M_{\lambda\mu}$ be the number of positive integer matrices with row sums $\mu$ & column sums $\lambda$.

then $$M_{\lambda\mu} = \sum_{\nu \geq \lambda, \mu} K_{\nu\lambda} \cdot K_{\nu\mu}.$$

In fact, there is an algorithmic correspondence between such matrices and pairs $(P, Q)$ of SSYT where $P$ has weight $\lambda$, $Q$ has weight $\mu$ and $P$ & $Q$ have the same shape (which is some $\nu \leq \lambda, \mu$).

Proving this theorem (& showing the algorithm) is the goal for the remainder of this presentation.

# The Shadow Path

- Let $A$ be a matrix with nonnegative integer entries. The *shadow* of the $(i,j)$ entry is all of the entries $(i',j')$ with $i' \geq i$, $j' \geq j$

The shadow defines a partial order:

$(i,j) \geq (i',j')$ if $(i',j')$ is in the shadow of $(i,j)$.

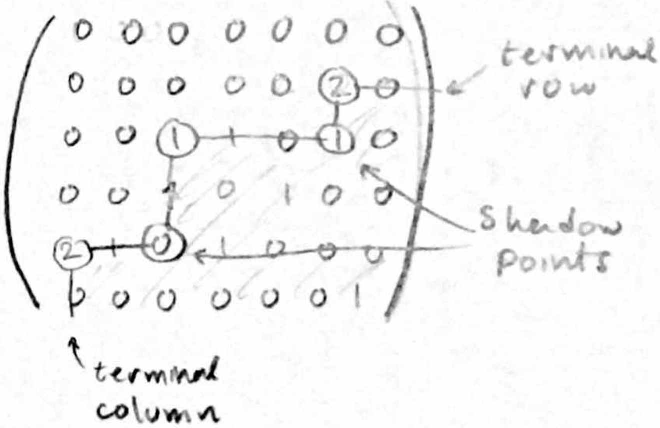
<u>A max'l entry</u> is a nonzero entry which is max'l wrt this order.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 6 & 0 \end{pmatrix}$$

We can arrange the maximal entries to have increasing row numbers, and in this order they'll also have decreasing column numbers. The zigzag path obtained by joining the max'l entries is the shadow path of $A$.

The column & row of the ends of the path are the "terminal column" & "terminal row".

The vertices which were not max'l entries of $A$ are "shadow points"




## Algorithm for Generating a Row (AROW):

Start with $A$, and let $S$ be the zero matrix of the same dimensions as $A$. Let $p$ & $q$ be empty strings which will become rows.

while $A \neq 0$, construct the shadow path of $A$. append the terminal column number to $p$ & the terminal row number to $q$. Subtract 1 from each max'l entry of $A$ and add 1 to each entry of $S$ corresponding to a shadow point in $A$.

return $S$, $p$, and $q$ when $A = 0$.

## AROW Example:

$$\begin{pmatrix}1&0&2\\0&2&0\\1&0&0\end{pmatrix} \longrightarrow \begin{pmatrix}0&0&2\\0&2&0\\1&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \quad 1 \quad 1$$

$A$ (input) $\qquad A \qquad S \qquad p \qquad q$

$$\longrightarrow \begin{pmatrix}0&0&1\\0&1&0\\0&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad 11 \quad 11$$

$A \qquad S \qquad p \qquad q$

$$\longrightarrow \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&0\\0&0&2\\0&1&0\end{pmatrix} \quad 112 \quad 111$$

$A \qquad S \qquad p \qquad q$ (output)

## Viennot-RSK algorithm (VRSK):

Start with A. Let P and Q be empty SSYT.

While $A \neq 0$, apply AROW to A which outputs $S$, $p$, and $q$.

Replace A by S, append p to P and q to Q as new rows of the SSYTs.

When $A = 0$, return P and Q.

eg
$$\underset{A}{\begin{pmatrix}1&0&2\\0&2&0\\1&0&0\end{pmatrix}} \rightarrow \underset{A}{\begin{pmatrix}0&0&0\\0&0&2\\0&1&0\end{pmatrix}} \ \underset{P}{\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}} \ \underset{Q}{\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}} \longrightarrow \underset{A}{\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}} \ \underset{P}{\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}} \ \underset{Q}{\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array}} \longrightarrow \underset{P}{\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}} \ \underset{Q}{\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}}$$

Note: this process is reversible.

Note: If the row sums / column sums are not decreasing, then neither will be the weights of P & Q (the weights are equal to the row/column sums.

## Why the VRSK algorithm works:

Say $A \geq B$ if $A - B$ has nonnegative entries.
Let $L(A)$ be the matrix with a 1 where $A$ has a max'l entry, 0 elsewhere.
Let $A^\circ = A - L(A)$. The sequence of matrices involved in AROW is $A \geq A^\circ \geq A^{\circ\circ} \geq \cdots \geq A^{(i)} \geq \cdots \geq 0$.

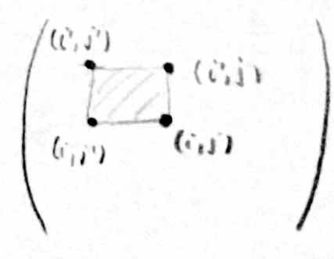
**Prop 1**: $P$ & $Q$ generated by VRSK have weakly increasing rows.

pf If $A \geq B$, the first nonzero row of $B$ is not above that of $A$, and the first nonzero column of $B$ is not to the left of that of $A$. in AROW we have $A \geq A^\circ \geq A^{\circ\circ} \geq \cdots$, so the terminal row # & terminal column # increases weakly w/ each iteration.

**Prop 2**: $P$ & $Q$ generated by VRSK have strictly increasing columns.

**Lemma 1**: $L(S(A)) \geq S(L(A))$ where $S(A)$ is the shadow matrix of $A$.

pf: first, $S(L(A))$ contains only 0s & 1s because $L(A)$ contains only 0s & maximal entries which are 1s, so no two entries can inhabit the same row or column. (also, $L(A)$ dies to one shadow path so $S(L(A)) \ni$ these shadow points)
Suppose $(i,j)$ is a nonzero entry of $S(L(A))$. Then $\exists$ maximal entries $(i',j)$ and $(i,j')$ s.t. $i' < i$ and $j' < j$. Also, $A$ has no max'l entries in $[i',i] \times [j',j]$. Now if the $(i,j)$ entry of $L(S(A))$ is zero, then $\exists$ $(k,\ell)$ s.t. $(i,j)$ is in $(k,\ell)$'s shadow in $S(A)$.




then $\exists$ nonzero entries $(k',\ell)$ and $(k,\ell')$ in $A$ which generate $(k,\ell)$ as a shadow point. But these entries would have to shadow $(i',j)$ or $(i,j')$ since they cannot lie in $[i',i] \times [j',j]$, which is a contradiction to the maximality of $(i',j)$ and $(i,j')$.

Lemma 2: $S(A^\circ) \geq S(A)^\circ$

Pf $S(A)^\circ = S(A) - L(S(A)) \leq S(A) - S(L(A)) = S(A^\circ)$.

Lemma 3: $S(A^{(i)}) \geq S(A)^{(i)} \quad \forall\, i > 0$.

Pf induction. Base case is Lemma 2. Suppose it's true for $i-1$. Then

$$S(A^{(i)}) = S(A^{(i-1)\circ}) \geq S(A^{(i-1)})^\circ \geq S(A)^{(i-1)\circ} = S(A)^{(i)}.$$

Proof of Propn 2: The $i^{th}$ entry of the first row of $P$ is the first nonzero column of $A^{(i)}$, while the $i^{th}$ entry of the second row of $P$ is the first nonzero column of $S(A)^{(i)}$. Since $S(A)^{(i)} \leq S(A^{(i)})$, this cannot come before the first nonzero column of $S(A^{(i)})$, which is strictly less than the FNZC of $A^{(i)}$ by the shadow construction.

Propn 3: If $(P,Q) = VRSK(A)$ and $A$ is $\lambda \times \mu$ (column sum, row sum) then $P$ has weight $\mu$ and $Q$ has weight $\lambda$.

proof: Let $R_i(A,S,p,q) = r_i(A) + r_i(S) + n_i(q)$ for $A,S$ matrices & $p,q$ rows of integers

$C_j(A,S,p,q) = c_j(A) + c_j(S) + n_j(p)$

where $r_i, c_j$ are the $i^{th}$ Row sum & $j^{th}$ column sum functions, and $n_i$ is the number of occurrences of $i$.

If $(A,S,p,q) \longrightarrow (A',S',p',q')$ in one step of AROW then $R_i(A,S,p,q) = R_i(A',S',p',q')$ and same for $C_j$. The row sum of the terminal row goes down by 1, but the row number gets appended to $q$. for other rows with more entries, $r_i(A) + r_i(S) = r_i(A') + r_i(S)$ since a 1 gets moved over from A to S.

So $(A,0,\emptyset,\emptyset) \xrightarrow{VRSK} (0,0,P,Q)$ conserves both $R_i$ and $C_j$ (applied to whole SSYT instead of just rows), so $n_i(Q) = r_i(A)$, and $n_j(P) = c_j(A)$.