Basic Representation Theory Stuff

Def A representation of G is (ρ, V) where V is a vector space over some field and $\rho : G \longrightarrow GL(V)$ is a gp hom.

Examples: $G = S_n$. $\mathbb{C}[S_n] = \mathbb{C}\text{-span}\{\delta_\sigma : \sigma \in S_n\} = \mathrm{Fun}(S_n, \mathbb{C})$ where $\delta_\sigma(\tau) = \delta_{\sigma \tau}$.

This is the "regular representation."

- Let $\lambda \vdash n$. An ordered partition of $\{1, \dots, n\}$ of shape λ is a collection of sets S_i s.t. $\{1, \dots, n\} = \sqcup S_i$ and $|S_i| = \lambda_i$.

  Let X_λ be the set of all such ordered partitions. $S_n \circlearrowright X_\lambda$.

  $\mathbb{C}[X_\lambda] = \mathrm{Fun}(X_\lambda, \mathbb{C})$ is the partition representation.

  (Note that if $G \circlearrowright X$ then $\mathbb{C}[X]$ is a repn of G with $\rho(g)(1_x) = 1_{g \cdot x}$, or $(\rho(g)f)(x) = f(g^{-1} \cdot x)$)

  Observe that $\mathbb{C}[X_{(1,\dots,1)}] \cong \mathbb{C}[S_n]$.

Defs:
- $W \subseteq V$ is an invariant subspace if $\rho(g)(W) \subseteq W \quad \forall g \in G$.
- A repn is simple if it has no nontrivial invariant subspaces

Def Let (ρ_1, V_1) and (ρ_2, V_2) be repns of G. A linear transformation $T : V_1 \longrightarrow V_2$ is called an intertwiner (or G-hom-sm) if The following diagram commutes $\forall g \in G$:

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "\rho_1(g)"'] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "T"] & V_2 \end{tikzcd}$$

The space of all intertwiners is denoted $\mathrm{Hom}_G(V_1, V_2)$.
Also, $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V)$.

Lemma (Schur): If V is a finite-dimensional simple repn of G over $\mathbb{C}$ then $\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot id_V$. i.e., $\dim \mathrm{Hom}_G(V, V) = 1$.

Lemma (Schur 2): If V_1 & V_2 are simple, $\dim \mathrm{Hom}_G(V_1, V_2) = 1$ or 0, depending on whether or not $V_1 \cong V_2$. i.e., $V_1 \cong V_2$ or $\dim \mathrm{Hom}_G(V_1, V_2) = 0$.

Proof of Schur 1: Any self-intertwiner $T: V \to V$ has an eigenvalue, say λ. Now $T - \lambda\, id_V$ is also an intertwiner, and it has nonzero kernel. But $\mathrm{Ker}(T - \lambda\, id_V)$ is an invariant subspace, so it must be all of V. Thus $T = \lambda\, id_V$.

Proof of Schur 2: If T is an intertwiner $V_1 \to V_2$, then $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ are invariant subspaces of V_1 & V_2.

__Defn:__ A finite-dimensional repn V is completely reducible if

$$V = \bigoplus V_i^{n_i} \quad \text{where each } V_i \text{ is simple and } V_i \not\cong V_j \text{ if } i \neq j.$$

n_i is called the multiplicity of V_i in V.

__Remark:__ By the Schur lemmas, ~~dim~~ $\dim \mathrm{Hom}_G(V_i, V) = n_i$.

__Theorem (Maschke):__ If V is a finite-dimensional repn of G over $\mathbb{C}$, then V is completely reducible.

__Note:__ If V & W are finite-dim repns of G over $\mathbb{C}$, $\dim \mathrm{Hom}_G(V,W) = \dim \mathrm{Hom}_G(W,V)$.

(This is because intertwiners are big matrices & so these dimensions are determined by the multiplicities of simple repns in V & W in a symmetric formula: if $V = \bigoplus V_i^{n_i}$ and $W = \bigoplus V_i^{m_i}$ then

$$\dim \mathrm{Hom}_G(V,W) = \sum_i m_i n_i \;)$$

__Theorem:__ If G is finite, then there are finitely many nonisomorphic ~~finite dim simple~~ finite-dimensional simple repns of G over $\mathbb{C}$.

__Proof:__ Let $\{V_\lambda\}$ be a set of non-isomorphic finite simple repns of G over $\mathbb{C}$. For each λ, $n_\lambda^* = \dim \mathrm{Hom}_G(V_\lambda, \mathbb{C}[G]) = \dim \mathrm{Hom}_G(\mathbb{C}[G], V_\lambda)$. But for any repn W, $\mathrm{Hom}_G(\mathbb{C}[G], W) \cong W$ because an intertwiner here is determined by where it sends δ_e. So $n_\lambda = \dim V_\lambda$. But $\mathbb{C}[G]$ is finite-dimensional, so $\{V_\lambda\}$ must be finite.

$*\, n_\lambda$ for $\mathbb{C}[G]$

Note: We have also just proved that $\mathbb{C}[G] = \bigoplus V_i^{\dim V_i}$ as V_i varies across all non-isomorphic simple finite-dim repns of G.

Thm (Intertwining Number Theorem): Let $X \circlearrowleft G \circlearrowright Y$, $|X|, |Y| < \infty$. Then $\dim \operatorname{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = |G \backslash (X \times Y)|$ where $G \circlearrowright (X \times Y)$ by diagonal action: $g \cdot (x, y) = (g \cdot x, g \cdot y)$.

To prove this, we need some notation/definitions/notes:

Note: any function $\kappa : X \times Y \to \mathbb{C}$ gives rise to a linear transformation $T_\kappa : \mathbb{C}[X] \to \mathbb{C}[Y]$ defined by $(T_\kappa f)(y) = \sum_{x \in X} \kappa(x, y) \cdot f(x)$.

When is T_κ an intertwiner? i.e. when do we have $\rho_Y(g)^{-1} \circ T_\kappa \circ \rho_X(g) = T_\kappa$ for all $g \in G$? for which κ? Expanding the LHS we get:

$$
\begin{aligned}
(\rho_Y(g)^{-1} \circ T_\kappa \circ \rho_X(g) f)(y) &= (T_\kappa \circ \rho_X(g) f)(g \cdot y) \\
&= \sum_{x \in X} \kappa(x, g \cdot y)\, (\rho_X(g) f)(x) \\
&= \sum_{x \in X} \kappa(x, g \cdot y)\, f(g^{-1} \cdot x) \\
&= \sum_{x \in X} \kappa(g \cdot x, g \cdot y)\, f(x)
\end{aligned}
$$

So we must have $\kappa(x, y) = \kappa(g \cdot x, g \cdot y)$ $\forall g \in G$. Thus κ must be constant on each G-orbit of $X \times Y$. Since every linear transformation $\mathbb{C}[X] \to \mathbb{C}[Y]$ has this form (is obtained from some $\kappa : X \times Y \to \mathbb{C}$ in this fashion), this proves the intertwining number theorem.

Snap back to reality...

Recall that $M_{\lambda\mu} = \#\{(a_{ij}) \in M_{n\times n}(\mathbb{Z}_{\geq 0}) : \begin{smallmatrix}\sum_j a_{ij} = \lambda_i \\ \sum_i a_{ij} = \mu_j\end{smallmatrix}\}$.

Recall that $S_n \circlearrowright X_\lambda$, and now $S_n \circlearrowright (X_\lambda \times X_\mu)$.

Theorem: $|S_n \backslash (X_\lambda \times X_\mu)| = M_{\lambda\mu}$. In fact, There is a bijection between the S_n-orbits of $X_\lambda \times X_\mu$ and the $\lambda \times \mu$ matrices.

Proof: Let $S = (S_1, \dots, S_\ell)$ and $T = (T_1, \dots, T_m)$ be elements of X_λ and X_μ.

define $r_{ij}(S,T) = |S_i \cap T_j|$. Let $r(S,T) = (r_{ij}(S,T))$.

Observe that $\sum_j r_{ij}(S,T) = \sum_j |S_i \cap T_j| = |S_i| = \lambda_i$, and similarly for $\sum_i r_{ij}(S,T) = \mu_j$, so $r(S,T)$ is an $\lambda \times \mu$ matrix.

- And, $\forall g \in S_n$, $|S_i \cap T_j| = |g(S_i \cap T_j)| = |gS_i \cap gT_j|$ so r descends to a well-defined function on $S_n \backslash (X_\lambda \times X_\mu)$.
- This function is injective: if $(S,T), (S',T') \in X_\lambda \times X_\mu$, satisfying $|S_i \cap T_j| = |S_i' \cap T_j'|$, Then $\bigsqcup_{i,j} S_i \cap T_j$ and $\bigsqcup_{i,j} S_i' \cap T_j'$ are Partitions of $\{1, \dots, n\}$ of the same shape, So there is a permutation $g \in S_n$ such that $g(S_i \cap T_j) = S_i' \cap T_j'$. Then also $g(S,T) = (S',T')$ so $(S,T) = (S',T') \mod S_n$.
- This function is also surjective: suppose (a_{ij}) is a $\lambda \times \mu$ matrix. Let $\{1, \dots, n\} = \bigsqcup_{ij} A_{ij}$ for any A_{ij} satisfying $|A_{ij}| = a_{ij}$. Now define $S_i = \bigsqcup_j A_{ij}$ and $T_j = \bigsqcup_i A_{ij}$. Then $r(S,T) = (a_{ij})$.

Combinatorial Resolution Theorem: Suppose $(P, \leq)$ is a finite partially ordered set, and $\{U_\lambda\}_{\lambda \in P}$ is a family of completely reducible representations of a group G. Let $M_{\lambda\mu} = \dim \mathrm{Hom}_G(U_\lambda, U_\mu)$. If there exist nonnegative integers $K_{\mu\lambda}$ for all $\mu \geq \lambda$ in P such that $K_{\lambda\lambda} = 1$ for each $\lambda \in P$ and

$$M_{\lambda\mu} = \sum_{\nu \geq \lambda, \mu} K_{\nu\lambda} \cdot K_{\nu\mu} \quad \text{for all } \lambda, \mu \in P,$$

Then, for every $\mu \in P$, There is a simple representation V_μ such that $U_\lambda = \bigoplus_{\mu \geq \lambda} V_\mu^{K_{\mu\lambda}}$ for all $\lambda \in P$.

Proof: Induction on $|P|$. Let λ_0 be a maximal element of P. Then $M_{\lambda_0\lambda_0} = K^2_{\lambda_0\lambda_0} = 1$, so $V_{\lambda_0} := U_{\lambda_0}$ is a simple repn. Also, $M_{\lambda_0\lambda} = K_{\lambda_0\lambda}$ for each $\lambda \in P$, so the multiplicity of V_{λ_0} in U_λ is $K_{\lambda_0\lambda}$. Thus there are representations U_λ^0 with no V_{λ_0} satisfying $U_\lambda = U_\lambda^0 \oplus V_{\lambda_0}^{K_{\lambda_0\lambda}}$.

Let $P^0 = P \setminus \{\lambda_0\}$. Let $M^0_{\lambda\mu} = \dim \mathrm{Hom}_G(U^0_\lambda, U^0_\mu)$ for all $\lambda, \mu \in P^0$. Then

$$\begin{aligned} M^0_{\lambda\mu} &= M_{\lambda\mu} - K_{\lambda_0\lambda} K_{\lambda_0\mu} \\ &= \sum_{\lambda_0 > \nu \geq \lambda, \mu} K_{\nu\lambda} \cdot K_{\nu\mu}, \end{aligned}$$

So $\{U^0_\lambda\}_{\lambda \in P}$ is a smaller collection of representations of G satisfying the hypotheses of the theorem.

Now recall the RSK correspondence: if $P = \{\lambda \vdash n\}$, $G = S_n$, and $U_\lambda = \mathbb{C}[X_\lambda]$, Then the numbers

$$K_{\mu\lambda} = \text{the number of SSYT of shape } \mu \text{ and weight } \lambda$$

Satisfy the hypotheses of The CRT. Thus we have Proved <u>Young's Rule</u>: $\forall \lambda \vdash n$, there exists a unique Simple representation V_λ of S_n such that

$$\mathbb{C}[X_\lambda] = \bigoplus_{\nu \geq \lambda} V_\nu^{K_{\nu\lambda}} .$$

<u>Def</u>: A standard young tableau (SYT) of shape λ is an SSYT of shape λ and weight $\mu = (1, \ldots, 1)$.

Let $f_\lambda = K_{\lambda(1,\ldots,1)}$, the number of SYT of shape λ.

Recall that $\mathbb{C}[X_{(1,\ldots,1)}]$ is the regular representation $\mathbb{C}[S_n]$, so as a special case of Young's Rule, we have

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} V_\lambda^{f_\lambda} .$$

So $\{V_\lambda\}_{\lambda \vdash n}$ is a complete collection of irreducible finite-dim representations of S_n over $\mathbb{C}$, and, moreover, $\dim V_\lambda = f_\lambda$