Zigzag numbers

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0 - L'Histoire

- Désiré André (1840-1917, France)
  - Chairman (and first treasurer) of Société Mathématique de France.
  - Worked with Catalan numbers and other combinatorial objects. Zigzag numbers (AKA alternating permutations) are similar in some aspects to catalan numbers.
  - Published <u>Sur les permutations Alternées</u> in 1881 in "Journal de mathématiques pures et appliquées.

I - Définition Des Permutations Alternées.

<u>Def</u> Alternating permutation: A permutation of $\{1, \dots, n\}$, σ, (André used $\{\alpha_1, \dots, \alpha_n\}$ and imposed an ordering $\alpha_i < \alpha_j$ iff $i < j$) so that $\sigma(i) < \sigma(i+1)$ and $\sigma(i) < \sigma(i-1)$ or $\sigma(i) > \sigma(i+1)$, $\sigma(i) > \sigma(i-1)$ for all $1 < i < n$. This is not the notation André used.

<u>Thm</u> There are the same number of alternating permutations which begin with an increase $(\sigma(1) < \sigma(2))$ as begin with a decrease $(\sigma(1) > \sigma(2))$

<u>Pf</u> 

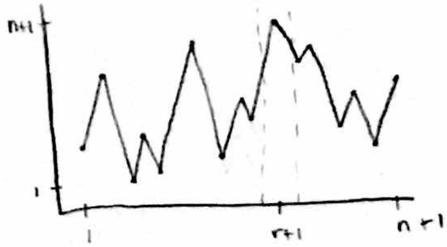
for $n \geq 2$. So, call the number of alternating permutations $2A_n$ for $n \geq 2$. A_n is the number of alternating permutations of n elements which begin with an increase (equiv. decrease).

Note $A_0 = 1$, $A_1 = 1$, $A_2 = 1$, $A_3 = 2$, $A_4 = 5$.

II - Calcul Des Nombres A_n.

Consider the either increasing or decreasing zigzag permutation on $n+1$ elements which has $n+1$ in the $(r+1)$st place. to the left there are r elements, and to the right there are $n-r$ ~~where $r+s=n$~~. the left r numbers can be chosen to be any alternating permutation which ends in a decrease, and the left $n-r$ can be any rising alt. perm.

II cont.



There are A_r ways to choose the zigzag permutation on these particular r elements, and A_{n-r} ways to choose the permutation on the s elements.

Also, there are $\binom{n}{r}$ ways to pick the r elements to the right of n+1, and there are n+1 places the n+1 could be, so we get the recurrence

$$2A_{n+1} = \binom{n}{0}A_0A_n + \binom{n}{1}A_1A_{n-1} + \binom{n}{2}A_2A_{n-2} + \cdots + \binom{n}{n}A_nA_0$$

André applied this formula a few times and found that

$A_5 = 16,\ A_6 = 61,\ A_7 = 272,\ A_8 = 1385,\ A_9 = 7936$

III - Fonction génératrice des fractions $\frac{A_n}{n!}$

Let $a_n = \frac{A_n}{n!}$. Since $\binom{n}{i} = \frac{n!}{(n-i)!\,i!}$, the above recurrence is transformed to:

$$2(n+1)\,a_{n+1} = a_0a_n + a_1a_{n-1} + a_2a_{n-2} + \cdots + a_na_0$$

Since $2(n+1)a_{n+1} = 2(n+1)\frac{A_{n+1}}{(n+1)!} = 2\frac{A_{n+1}}{n!} = \frac{1}{n!}\sum_{i=0}^{n}\frac{n!}{(n-i)!\,i!}A_iA_{n-i} = \sum_{i=0}^{n}\frac{A_i}{i!}\frac{A_{n-i}}{(n-i)!} = \sum_{i=0}^{n} a_ia_{n-i}$.

for $n \geq 2$.

Note that the number of zigzag permutations is less than or equal to the number of permutations, namely $2A_n \leq n!$ so $a_n \leq \frac{1}{2}$.

Thus the series $Y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ converges absolutely for $x \in (0,1)$.

So we can square it to obtain $Y^2 = a_0^2 + (a_0a_1 + a_1a_0)x + (a_0a_2 + a_1a_1 + a_2a_0)x^2 + \cdots$

So $Y^2 = a_0^2 + 2\cdot2\cdot a_2x + 2\cdot3\cdot a_3x^2 + 2\cdot4\cdot a_4x^3 + \cdots$

But also $\frac{dY}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$ $\Big\}$ so $Y^2 = a_0^2 + 2\left(\frac{dY}{dx} - a_1\right)$

and since $a_0 = a_1 = 1$, we have $Y^2 = 2\frac{dY}{dx} - 1$. André immediately recognized that $\arctan g\, Y = \frac{x}{2} + C$. To see this, Dörie notes that $\frac{\frac{dY}{dx}}{1+Y^2} - \frac{1}{2} = 0$

So that $\arctan g\, Y - \frac{1}{2}x$ is a constant function. Considering $x = 0$, we see that $Y = 1$, $\arctan g\, Y = \frac{\pi}{4}$, and so $C = \frac{\pi}{4}$. Thus $y = \tan g\left(\frac{\pi}{4} + \frac{x}{2}\right)$,

So $\tan g\left(\frac{\pi}{4} + \frac{x}{2}\right) = a_0 + a_1x + a_2x^2 + \cdots$ for all $x \in (-1, 1)$

$$= A_0 + A_1x + \frac{A_2}{2!}x^2 + \frac{A_3}{3!}x^3 + \cdots$$