Quadrature of the Hyperbola

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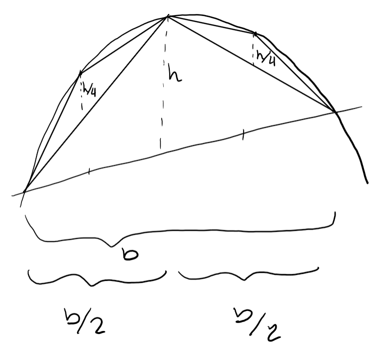
Quadrature: classical term for 'area.' The reason it's called quadrature is because to find the area of a region it suffices to find a quadrilateral with the same area.

Archimedes (287-212 BC) provides an ancient example: finding the quadrature of a section of a parabola:



Given a line sectioning off part of a parabola, we will find the area enclosed between the parabola and the line.

First, construct a triangle with its base being the section of the line enclosed in the parabola and its top vertex being the point on the parabola directly above the midpoint of the base.

Now construct more triangles that are enclosed within the new parapolic section created. Each time we construct a triangle, it's base is one of the sides of a previous triangle and it's top vertex is the point on the parabola that is directly above the midpoint of the base.

The first triangle has an area of $\frac{1}{2}bh$, and the second triangles each have an area of $\frac{1}{16}bh$ since their heights are $\frac{1}{4}$ of the height the original triangle and their widths are $\frac{1}{2}$ of the width of the original triangle. We could continue constructing smaller triangles for as long as we want. Each triangle, then, has an area that is $\frac{1}{8}$ of the area of the triangles in the layer below it, and there are twice as many triangles in a given layer as there are in the layer below it. Also, the sum of the areas of the triangles will never exceed the area of the parabolic section since the triangles are all contained in the section and do not overlap.

Archimedes used many results from Appolonius of Perga (262-190 BC) to show that the triangles in the second layer are $\frac{1}{8}$th of the triangle in the first layer.

We can see that the nth layer of triangles includes 2^n triangles and each triangle in the nth layer has an area of $\left(\frac{1}{8}\right)^n$ times the area of the original triangle. If we let A equal the area of the original triangle, we will see that the area of the entire parabolic section is

$$A + 2\frac{1}{8}A + 4\frac{1}{64}A + \cdots + 2^n\left(\frac{1}{8}\right)^n A + \cdots$$

Or, simplifying the expression,

$$A\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^n$$

And simplifying this geometric series, we see that the area of the entire parabolic section is $\frac{4}{3}$ of the area of the first triangle constructed in this manner. The area of the triangle is trivial to calculate, so this is a good way to calculate the area under a parabola.

Table of information

Name	**Life**	**Curve**
Archimedes	287 - 212 BC	x^2
Alhazen	965 - 1040	x^3 and x^4
Bonaventura Cavalieri	1598 - 1647	x^5 through x^9
John Wallis	1616 - 1703	x^n, n rational (including negatives)
Gregory of Saint-Vincent	1584 - 1667	x^{-1}

So Archimedes found a way to calculate the area under a quadratic curve. More than a thousand years later, an Arab mathematician named Alhazen (965-1040) was able to calculate the area under cubic and quartic curves. His method involved using induction to prove formulas for sums of integer cubes and fourth powers, and then (using a long and complicated proof) using these to find the volume of a section of a paraboloid, which is a similar problem to finding the area under a quartic curve.

Note It may have been the case that areas under cubic curves had already been calculated.

Now, as we know today, areas under curves of the form x^n are proportionally related to the curves of the form x^{n+1}. In other words (and using modern notation), we have the following equality (assuming this integral exists):

$$\int_a^b x^n\,dx = \frac{a^{n+1} - b^{n+1}}{n+1}$$

This is called Cavalieri's Quadrature formula, and it was discovered by Bonaventura Cavalieri (1598-1647) about 600 years after Alhazen. Cavalieri only actually proved this principle for n being an integer from 0 up to 9.

The mechanism of his proof relied on what Cavalieri is most known for: Cavalieri's Principle. Cavalieri's Principle states that the volumes (areas) of two objects are equal if their corresponding cross sections are in all cases equal. *(Draw a picture)*

Note Two cross sections correspond if they are equidistant from a chosen base (usually one end of the object).

A short while later, John Wallis (1616-1703) proved that Cavalieri's quadrature formula held for rational powers and negative powers (where the integral exists). Maybe a reason Wallis was able to make these generalizations was because, where Cavalieri only compared areas to areas and volumes to volumes (keeping the dimensions of the number in mind), Wallis simply considered them units and ignored the dimension. This is helpful because fractional and negative powers have no intuitive real-life volume or area correspondents.

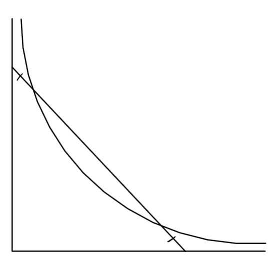
John Wallis attempted to generalize the rule (Cavalieri's quadrature formula) to $y = x^{-1}$, but was incorrect. I tried reading a translation of his "Arithmetica Infinitorum" (1656) and found the proof but I still couldn't really understand what he was doing.

Note Wallis knew that Cavalieri's principle would not hold for $n = -1$, but his method for overcoming this was incorrect.

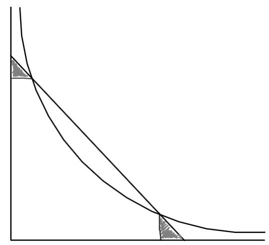
A few years before Wallis's "Arithmetica Infinitorum" was published, Gregory of Saint-Vincent (1584-1667) was able to find equations about the area under the curve $y = x^{-1}$, the rectangular hyperbola. Gregory of Saint-Vincent was a Jesuit mathematician and professor who lived in Belgium, he studied mathematics, philosophy, and theology. He started his teaching career by teaching greek, and then moved to teaching mathematics. During his early days as a professor he published books on optics, gravity, and comets.

This construction was published in Saint-Vincent's "Opus Geometricum Quadrature Circuli Sectionum Coni" (1647). This book (as the title suggests) claims to have foudn a way to square the circle with a method involving conic sections, and it was very long (over 1200 pages) However, nobody actually believed that squaring the circle was possible, and many mathematicians searched the book thoroughly for errors. Four years after the publishing of the book (in 1651), Christiaan Huygens (1629-1695) found a flaw in Saint-Vincent's argument, so the book had a bad reputation. But there were many correct propositions in this book, including the following, that areas under a hyperbolic curve are equivalent when the coordinates along the asymptotes under the sections are in geometric progression.

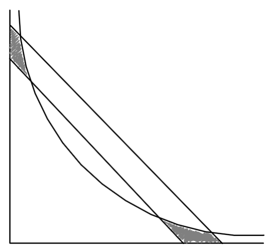
He began by examining the curve $xy = k$ for some constant k. Appolonius of Perga (262-190 BC) proved that if you draw a line intersecting the axes and the hyperbola, the segments between the axes and the intersection with the hyperbola will be of equivalent length:



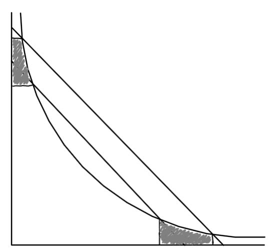
Using that fact, we can see that these two triangles are congruent (they are similar because they are both scaled versions of the larger triangle, and they are then congruent because their hypotenuses are equal.



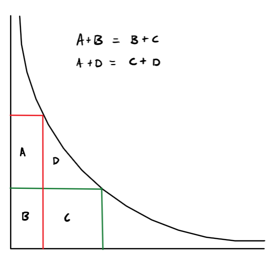
By Cavalieri's principle, using the fact that each line is equivalent to the opposite line, we see that these two regions are equivalent.



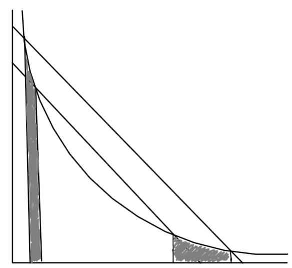
By Cavalieri's principle, using the fact that each line is equivalent to the opposite line, we see that these two regions are equivalent. And, since these two pairs of triangles are congruent, we see that these two regions are equivalent:



And, since $xy = k$, the area of each of these boxes is k. Therefore, $A = C$, so $A + D = C + D$.



Applying this to the previous picture, we see that the following two regions are equivalent:



Now, if one of the lines is tangent to the hyperbola, the x-coordinates of the intersection points of the two lines are in geometric progression. This can be seen as follows (the following stuff is my work, I'm not sure how Gregory of Saint-Vincent proved this).

1: Let the first line (y_1) have slope $-m$, and let it be tangent to the hyperbola. This means (since $y_1 = \frac{k}{x}$), $-m = y_1' = -\frac{k}{x^2}$, so $x = \sqrt{\frac{k}{m}}$.

2: Since $xy_1 = k$, $y = \sqrt{mk}$. Therefore, $y_1 = -mx + \sqrt{mk} + \sqrt{mk} = -mx + 2\sqrt{mk}$.

3: Now (since the two lines are parallel), $y_2 = y_1 + b = -mx + 2\sqrt{mk} + b$.

4: The points where this line intersects the hyperbola are where $-mx + 2\sqrt{mk} + b = \frac{k}{x}$, or where $mx^2 - (2\sqrt{mk} + b)x + k = 0$.

5: Using the quadratic formula, we see that in this case

$$\begin{aligned} x &= \frac{b + 2\sqrt{mk} \pm \sqrt{(b + 2\sqrt{mk})^2 - 4mk}}{2m} \\ &= \frac{b + 2\sqrt{mk} \pm \sqrt{b^2 + 4b\sqrt{mk}}}{2m} \end{aligned}$$

(Place these on diagram)

6: The greater of these divided by $\sqrt{\frac{k}{m}}$ is equal to $\sqrt{\frac{k}{m}}$ divided by the smaller of these. This can be seen as follows:

$$\begin{aligned} \frac{b + 2\sqrt{mk} + \sqrt{b^2 + 4b\sqrt{mk}}}{2m\sqrt{\frac{k}{m}}} \Big/ \frac{2m\sqrt{\frac{k}{m}}}{b + 2\sqrt{mk} - \sqrt{b^2 + 4b\sqrt{mk}}} &= \frac{(b + 2\sqrt{mk})^2 - b^2 - 4b\sqrt{mk}}{4m^2\frac{k}{m}} \\ &= \frac{b^2 + 4b\sqrt{mk} + 4mk - b^2 - 4b\sqrt{mk}}{4mk} \\ &= 1 \end{aligned}$$

So if we let the left endpoint equal 1 and the middle point equal r, we see that the right endpoint must equal r^2. So if we let $A(x)$ equal the area under the hyperbola from 1 to x, we see that $A(r^2) = A(r) + A(r)$ (recall that the two areas are equivalent).

In fact, even if one line isn't tangent to the hyperbola, if the lines intersect the hyperbola at four points $a < b < c < d$, then $\frac{b}{a} = \frac{d}{c}$. This can be shown with more algebraic manipulation.

Now, in modern notation, this means that (if we let $a = 1$), we have the following equality:

$$\int_1^a \frac{1}{x}\,dx = \int_b^{ab} \frac{1}{x}\,dx$$

$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx = \int_1^{ab} \frac{1}{x}\,dx$$

So, for any two numbers x and y, $A(xy) = A(x) + A(y)$.

While Gregory of Saint-Vincent proved that the x-coordinates of the intersection points lie in geometric progression, it was Alphonse Antonio de Sarasa (1617 - 1667) who applied this to his study of logarithms.

De Sarasa was attempting to solve a problem like this after being prompted by Mersenne (1588-1648): "Given three arbitrary magnitudes, rational or irrational, and given the logarithms of two, to find the logarithm of the third geometrically." Of course, the magnitudes in this case are hardly arbitrary; they are in geometric progression.

I'm not sure Sarasa (or anyone at the time) thought of logarithms of being some sort of inverse of exponentiation, or thought of them as having bases. But, when $k = 1$, we simply have the curve $y = x^{-1}$, for which the area under the curve is proportional to the natural logarithm.