Pólya's Theorem on Random Walks

Vilas Winstein

March 23, 2021

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"A drunk human will find their way home, but a drunk bird may get lost."

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Since the events of first returning on step n are disjoint, we have

$$p = \sum_{n \geq 0} p_n.$$

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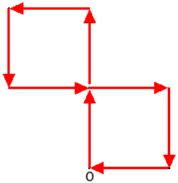

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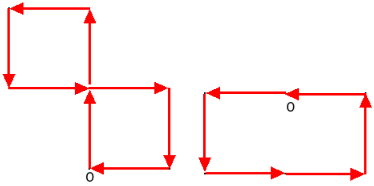

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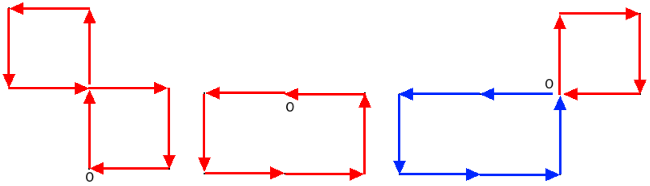

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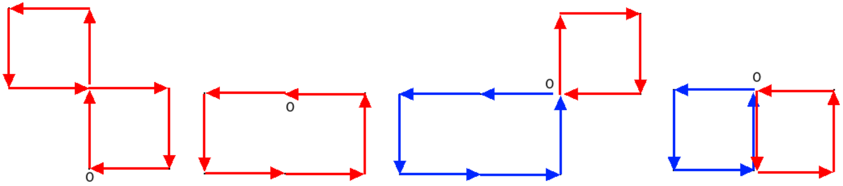

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The equation becomes

$$Q(z) - 1 = P(z)Q(z).$$

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$$P(1) = \sum_{n \geq 0} p_n 1^n = \sum_{n \geq 0} p_n = p.$$

So $P(z)$ converges at $z = 1$ as well, and we can take the limit from inside:

$$\lim_{\substack{z \to 1^- \\ z \in [0,1)}} P(z) = P(1) = p.$$

Solving the equation $P(z)Q(z) = Q(z) - 1$ for $P(z)$, this means that

$$p = \lim_{\substack{z \to 1^- \\ z \in [0,1)}} \left(1 - \frac{1}{Q(z)}\right) = 1 - \frac{1}{\lim_{\substack{z \to 1^- \\ z \in [0,1)}} Q(z)}.$$

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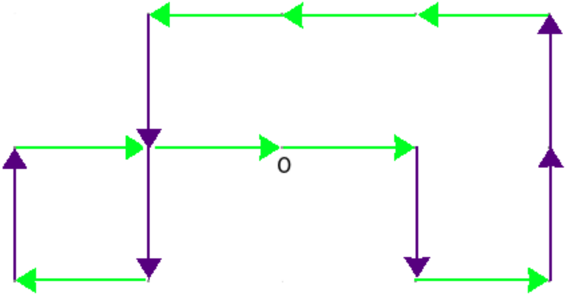

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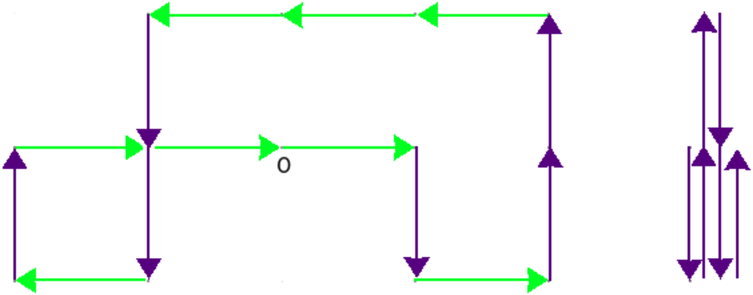

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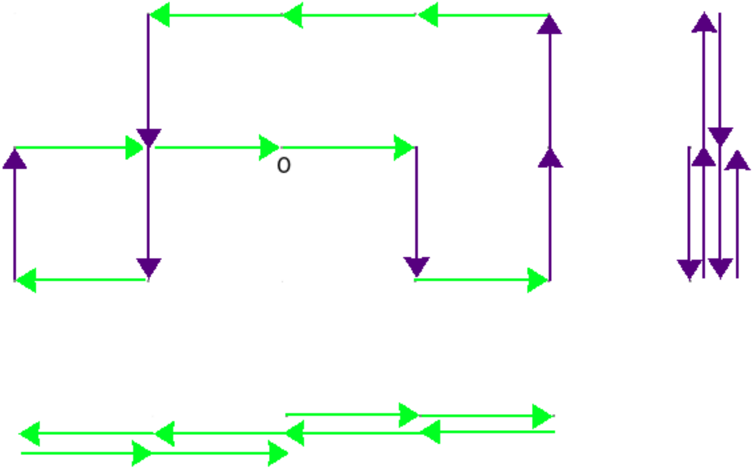

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Don't worry! The formulas that we will be using are a special case of these, and are a bit simpler.

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It will be handy to remember the integral expression for $I_0(z)$, which we will use later:

$$I_0(z) = \frac{1}{\pi} \int_0^{\pi} e^{z \cos\theta} \, d\theta.$$

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We will estimate this, as $t \to \infty$, using an asymptotic analysis method which is called *Laplace's method* (and which uses *Laplace's principle*).

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To quantify this, let's expand $f(\theta)$ using it's second Taylor polynomial approximation:

$$f(\theta) \approx f(0) - |f''(0)|\frac{\theta^2}{2}.$$

Laplace's Method

Now we can use that approximation to approximate the integral:

$$\int_0^{\pi} e^{tf(\theta)}\, d\theta \approx e^{tf(0)} \int_0^{\pi} e^{-t|f''(0)|\frac{\theta^2}{2}}\, d\theta.$$

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So, hopefully, we have the following good approximation:

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as $t \to \infty$, where $C = \frac{d^{d/2}}{2^{d/2}\pi^{d-d/2}}$ is a constant. Of course, C depends on d, but that doesn't matter to us at this point, since we are considering the convergence of an integral in t and a limit in z.

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Remember, we were trying to determine whether the integral limit

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So we have proved Pólya's theorem!

References

[1] J J O'Connor and E F Robertson. "George Pólya." MacTutor Biographies, November 2002. Web. 20 Mar. 2021.

[2] Novak, Jonathan. "Polya's random walk theorem." The American Mathematical Monthly 121.8 (2014): 711-716.

[3] Pólya, Georg. "Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz." Mathematische Annalen 84.1 (1921): 149-160.

[4] Wikipedia contributors. "George Pólya." Wikipedia, The Free Encyclopedia, 4 Mar. 2021. Web. 20 Mar. 2021.