## $\pi$ is irrational

born in Switzerland, eventually worked alongside Euler.

- Johann Heinrich Lambert (1728–1777) – Did a lot of math and physics, particularly optics & trigonometry (he was the first to use hyperbolic trigonometric functions). He gave what is widely recognized as the first rigorous proof of $\pi \notin \mathbb{Q}$, ~~[illegible]~~

[note: Lambert's proof is long & tedious so we'll be cherry-picking & black-boxing.]

## Continued fractions:

① $$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Simple ctd fraction ($a_i \in \mathbb{N}$, always converges)

② complicated ctd fraction:

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cdots}}$$

$a_i, b_i \in \mathbb{Z}$.

Convergence is tricky usually

③ Meaning of "convergence" for a continued frac.:

$$x_n = a_0 + \cfrac{b_1}{a_1 + \cfrac{\ddots}{\ddots + \cfrac{b_n}{a_n}}} \to x \text{ as } n \to \infty.$$

$n^{\text{th}}$ convergent.

④ with simple continued fractions, infinite $\iff$ irrational.

but with complicated ctd fs, it is not so simple.

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**Theorem 2:** The infinite continued fraction $\cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}}$ converges to an irrational value if $a_n > b_n + 1$ for all $n \geq$ some $n_0$. (where $a_i, b_i \in \mathbb{N}$)

**Proof:** Since $\frac{b_1}{a_1 + x}$ is irrational iff $x$ is irrational, it suffices to prove in the case $n_0 = 1$. The continued fraction converges to a value in $(0,1)$.

write this as a theorem! ← (non-trivial exercise of algebra involving numerators & denominators of continued fraction convergents).

Suppose that the value is $\frac{A_2}{A_1}$, where both are positive integers. Then $0 < A_2 < A_1$. Then

$$\frac{A_1}{A_2} = \cfrac{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}}{b_1} \Rightarrow \frac{A_2 a_1 - A_1 b_1}{A_2} = \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}$$

but $\cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots}}$ satisfies hypothesis of Theorem 1 as well, so, letting $A_3 = A_2 a_1 - A_1 b_1$ we find that $0 < A_3 < A_2 < A_1$. This can be continued indefinitely but all of these are positive integers. Absurd.

In the proof of Theorem 2, we can actually weaken the hypothesis to: $a_n \geq b_n + 1 \;\; \forall n \geq n_0$ and $a_n > b_n + 1$ for infinitely many $n$ (see details of Theorem 1).

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To start guessing at a continued fraction for $\tan(x)$, Lambert begins with the power series for sin & cos:

$$\sin(v) = \sum_{n=0}^{\infty} \frac{(-1)^n v^{2n+1}}{(2n+1)!}, \quad \cos(v) = \sum_{n=0}^{\infty} \frac{(-1)^n v^{2n}}{(2n)!}.$$

$$\tan(v) = \frac{v - \frac{v^3}{6} + \frac{v^5}{120} - \cdots}{1 - \frac{v^2}{2} + \frac{v^4}{24} - \cdots} = \cfrac{v}{\cfrac{1 - \frac{v^2}{2} + \frac{v^4}{24} - \cdots}{1 - \frac{v^2}{6} + \frac{v^4}{120} - \cdots}} = \cfrac{v}{1 - \left( \cfrac{\frac{v^2}{3} - \frac{v^4}{30} + \cdots}{1 - \frac{v^2}{6} + \frac{v^4}{120} - \cdots} \right)}$$

$$= \cfrac{v}{1 - \cfrac{v^2}{3\left( \cfrac{1 - \frac{v^2}{6} + \frac{v^4}{120} - \cdots}{1 - \frac{v^2}{10} + \cdots} \right)}} = \cfrac{v}{1 - \cfrac{v^2}{3 - \left( \cfrac{\frac{v^2}{5} - \cdots}{1 - \frac{v^2}{10} + \cdots} \right)}} = \cfrac{v}{1 - \cfrac{v^2}{3 - \cfrac{v^2}{5 - \cdots}}}.$$

So he conjectured that $\tan(v) = \cfrac{v}{1 - \cfrac{v^2}{3 - \cfrac{v^2}{5 - \cfrac{v^2}{7 - \cfrac{v^2}{9 - \cdots}}}}}$,

and then proved this by formalizing the previous calculations & creating a few recurrence relations.

$$\text{So} \quad \tan\left(\frac{\varphi}{\omega}\right) = \cfrac{\varphi}{\omega - \cfrac{\omega\varphi^2}{3\omega^2 - \cfrac{\omega^2\varphi^2}{5\omega^2 - \cfrac{\omega^2\varphi^2}{7\omega^2 - \cdots}}}} = \cfrac{\varphi}{\omega - \cfrac{\varphi^2}{3\omega - \cfrac{\varphi^2}{5\omega - \cdots}}}$$

So, if $\varphi$ and $\omega$ are both integers, we get a continued fraction of the form in the hypothesis of thm 2.

So if $v$ is rational and nonzero then $\tan(v)$ is irrational.

but $\tan(\frac{\pi}{4}) = 1$, so $\frac{\pi}{4}$ is irrational so $\pi$ is irrational.

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you may know that there is another continued fraction involving $\pi$: $\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}}$. However, there were no tools (like theorem 2) known at the time which could be used to show this is irrational.

→ this was due to Lord Brouncker in 1665, 103 years before Johann Lambert's proof that $\pi$ is irrational.

- Lambert's was the first rigorous proof of this fact (which was published) but it was a commonly held belief. Euler, one of Lambert's collaborators suspected $\pi \notin \mathbb{Q}$, → but could not prove it! and it is also speculated that the indian mathematician Aryabhata believed $\pi$ was irrational all the way back in 500 BC.
- And of course, the greeks also posed the question of "squaring the circle," which is very related.

$\pi$ was not proved to be transcendental until 1882 by Ferdinand von Lindemann, although Lambert conjectured that $\pi$ was transcendental in the paper we examined (even though the existence of transcendental numbers in general was not known at the time).

Theorem 1: If the infinite continued fraction $\cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cdots}}$ satisfies $a_n \geq b_n + 1 \ \forall n$, then it converges to a value $A \in (0,1]$.

if $a_n > b_n + 1$ for some $n$ then $A < 1$.

Proof? nah

More stuff that Lambert did in his paper (notice the "logarithmiques" in his title).

Theorem 3: The infinite continued fraction $\cfrac{b_1}{a_1 + \cfrac{b_1}{a_2 + \cdots}}$ converges to an irrational value if $a_n \geq b_n \ \forall n \geq n_0$.

Continued fraction for e: $\dfrac{e^u + 1}{2} = \cfrac{1}{1 - \cfrac{1}{\frac{2}{u} + \cfrac{1}{\frac{6}{u} + \cfrac{1}{\frac{10}{u} + \cfrac{1}{\frac{14}{u} + \cdots}}}}}$

So $e^u = -1 + \cfrac{2}{1 - \cfrac{u}{2 + \cfrac{u^2}{6 + \cfrac{u^2}{10 + \cfrac{u^2}{14 + \cdots}}}}}$ ← skip this one

So $e^{\frac{A}{B}} = -1 + \cfrac{2}{1 - \cfrac{A}{2B + \cfrac{A^2}{6B + \cfrac{A^2}{10B + \cfrac{A^2}{14B + \cdots}}}}}$

thus if $u \neq 0$ is rational, $e^u$ is irrational.

this fact had already been proven in the early 1700s by Johann Bernoulli, but by a different method.

and Euler too

$$e = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\ddots}}}}}}}}$$

(euler, 1737).

the wikipedia page for "irrationality of e" has in it's "generalizations" section that Liouville proved in 1840 that $e^2$ is irrational, and I could not find anything online saying explicitly that Lambert had the first proof of $e^{\mathbb{Q}} \not\subset \mathbb{Q}$, but I also couldn't find any earlier proof of this fact.

the proof taught today involves some strange polynomial functions but I like Lambert's proof better.

Another way to write tan(v) is

$$\tan(v) = \cfrac{1}{\cfrac{1}{v} - \cfrac{1}{\cfrac{3}{v} - \cfrac{1}{\cfrac{5}{v} - \cfrac{1}{\cfrac{7}{v} - \cdots}}}}$$

If we use hyperbolic trig...

$$\frac{e^{v} - e^{-v}}{2} = v + \frac{v^3}{3!} + \frac{v^5}{5!} + \frac{v^7}{7!} + \cdots$$

$$\frac{e^{v} + e^{-v}}{2} = 1 + \frac{v^2}{2!} + \frac{v^4}{4!} + \frac{v^6}{6!} + \cdots$$

$$\frac{e^{v} - e^{-v}}{e^{v} + e^{-v}} = \cfrac{1}{\cfrac{1}{v} + \cfrac{1}{\cfrac{3}{v} + \cfrac{1}{\cfrac{5}{v} + \cdots}}}$$

$$\|$$

$$\frac{e^{2v} - 1}{e^{2v} + 1}$$, donc

$$\frac{e^{x} - 1}{e^{x} + 1} = \cfrac{1}{\cfrac{2}{x} + \cfrac{1}{\cfrac{6}{x} + \cfrac{1}{\cfrac{10}{x} + \cdots}}}$$

and

$$\frac{e^{x} - 1}{e^{x} + 1} = \frac{e^{x} + 1 - 2}{e^{x} + 1} = 1 - \frac{2}{e^{x} + 1} \Rightarrow \frac{e^{x} + 1}{2} = \cfrac{1}{1 - \cfrac{1}{\cfrac{2}{x} + \cfrac{1}{\cfrac{6}{x} + \cdots}}}$$