The Mathematics of Light and Vision

Vilas Winstein

January 25, 2022

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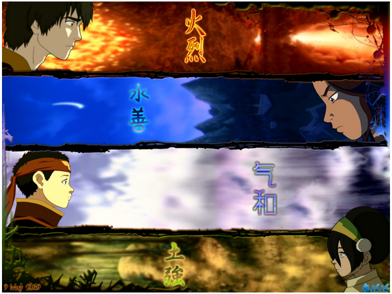

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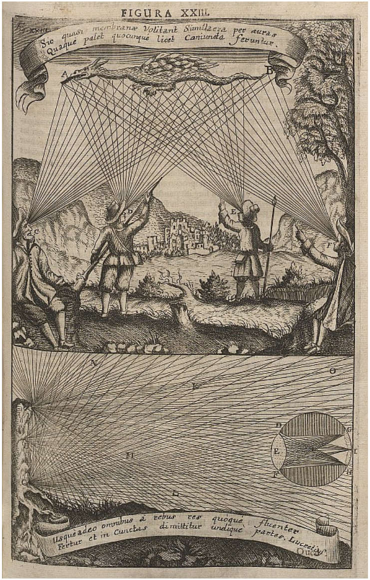

Emission Theory of Vision (Evidence)

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Computer Graphics

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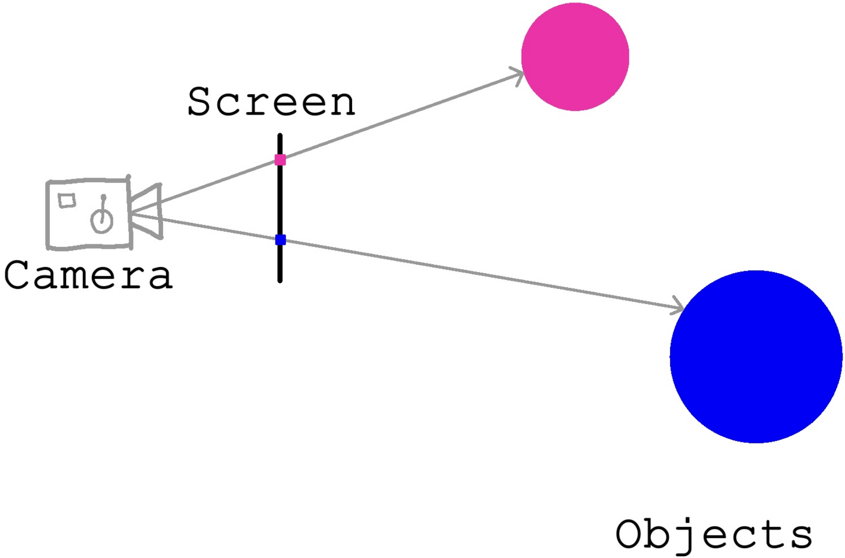

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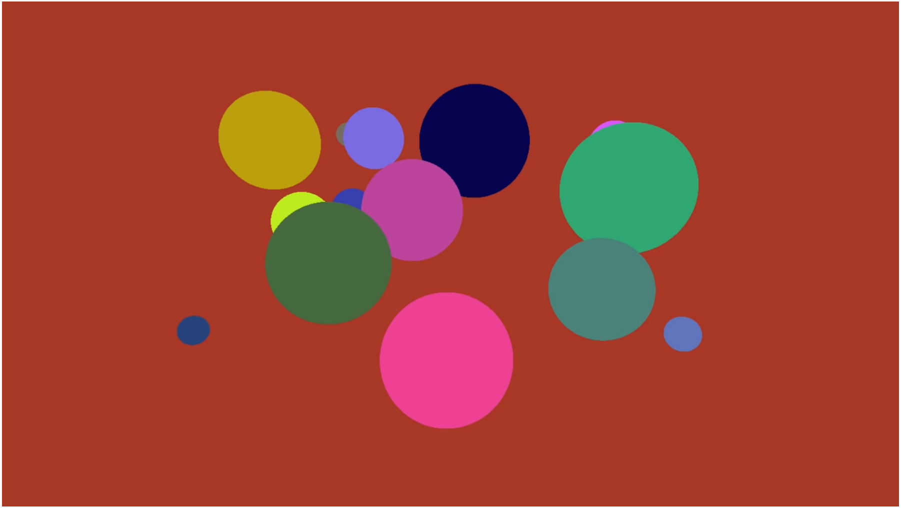

Reflection Theory of Vision

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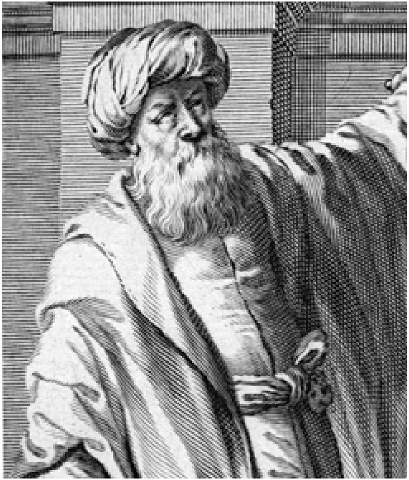
Ibn al-Haytham



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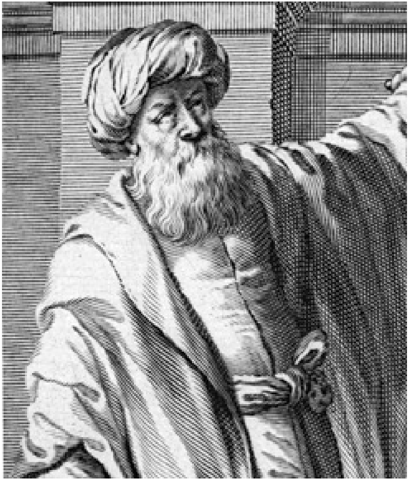

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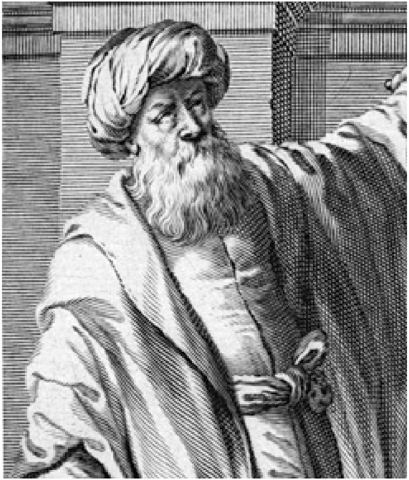

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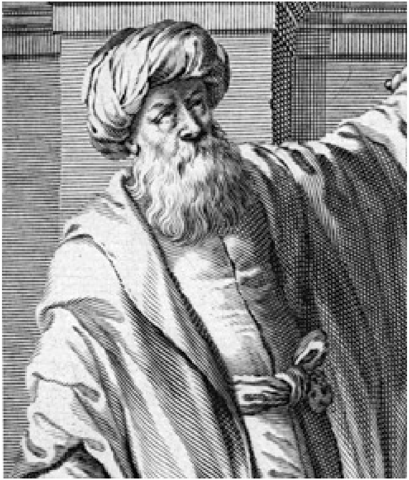

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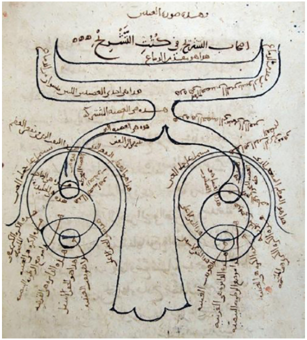

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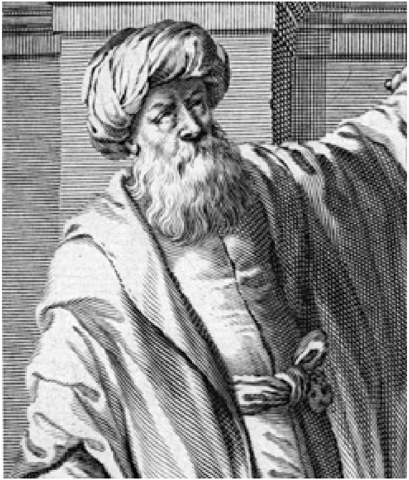

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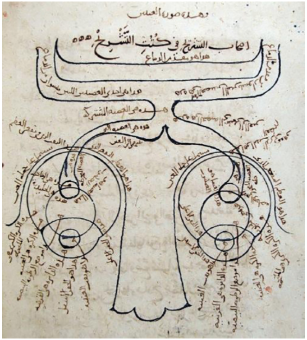

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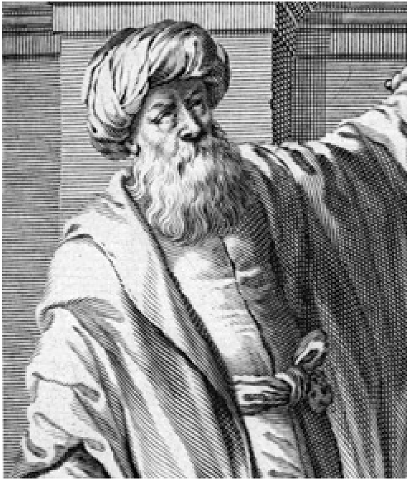

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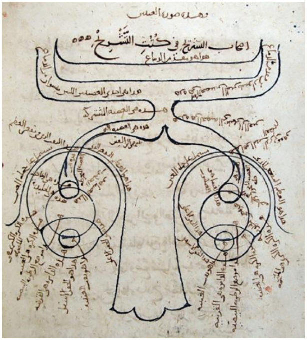

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"The brain processes visual signals from the eyes; this is how we see."

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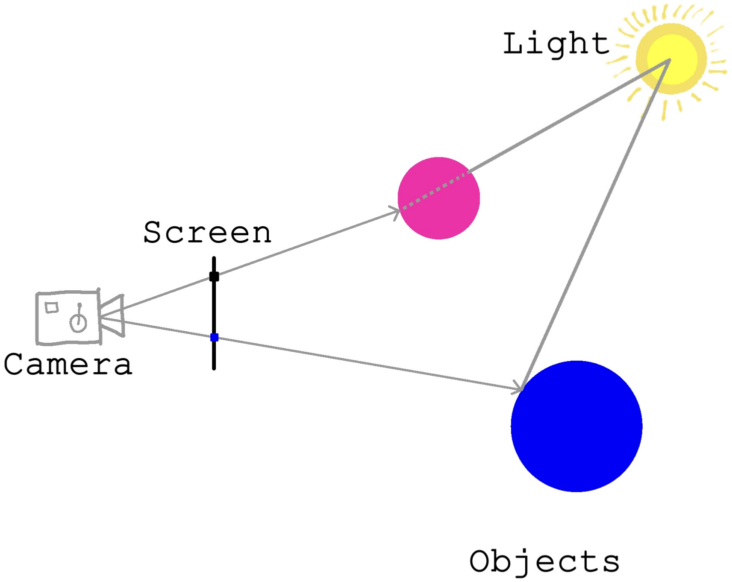

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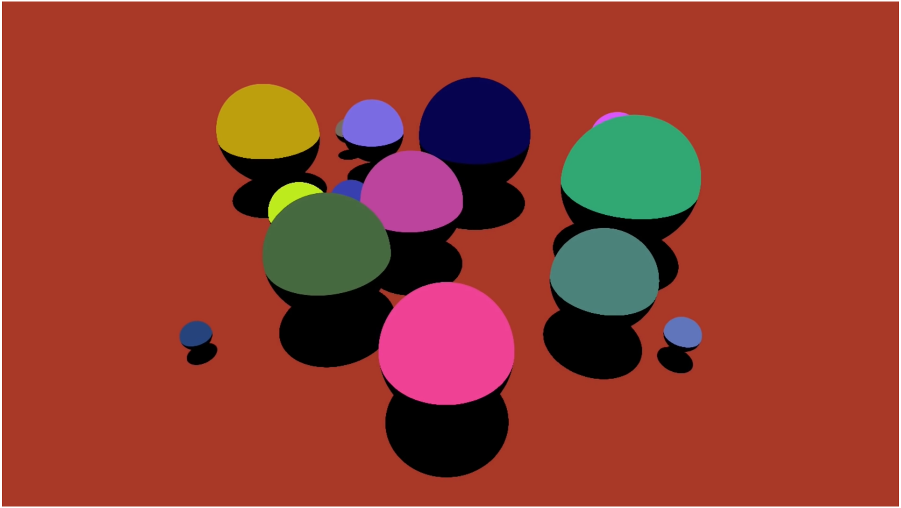
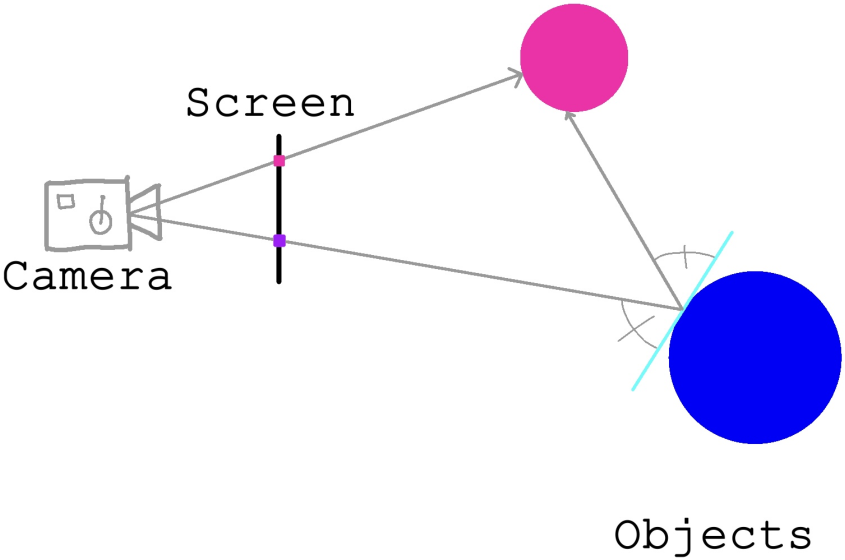
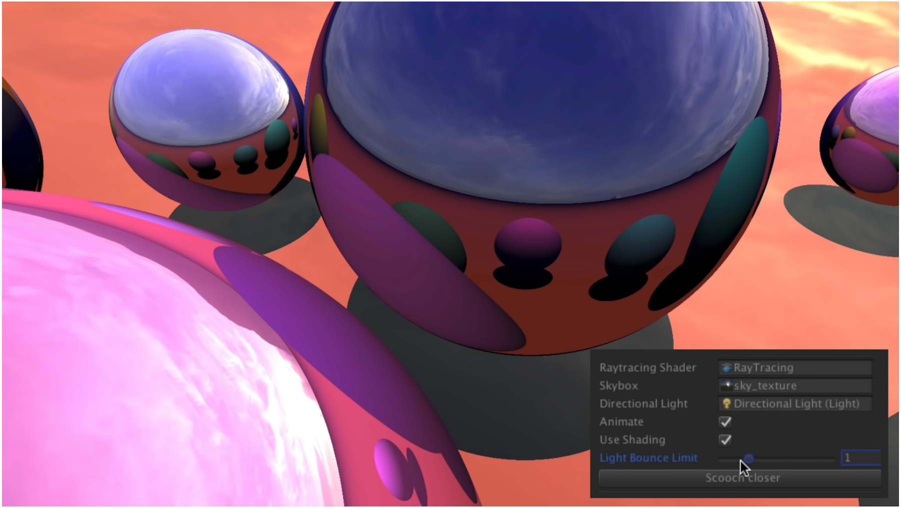



Screen
Camera
Objects

Computer Graphics (Reflections)



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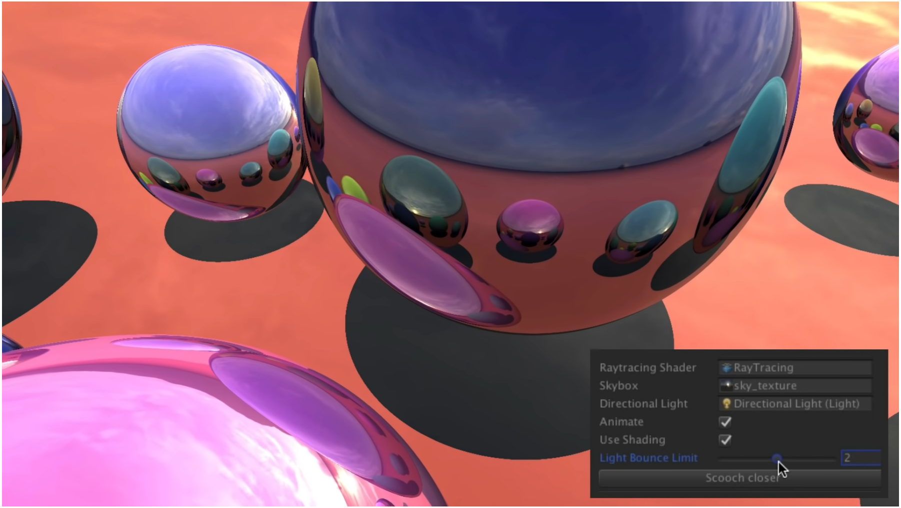

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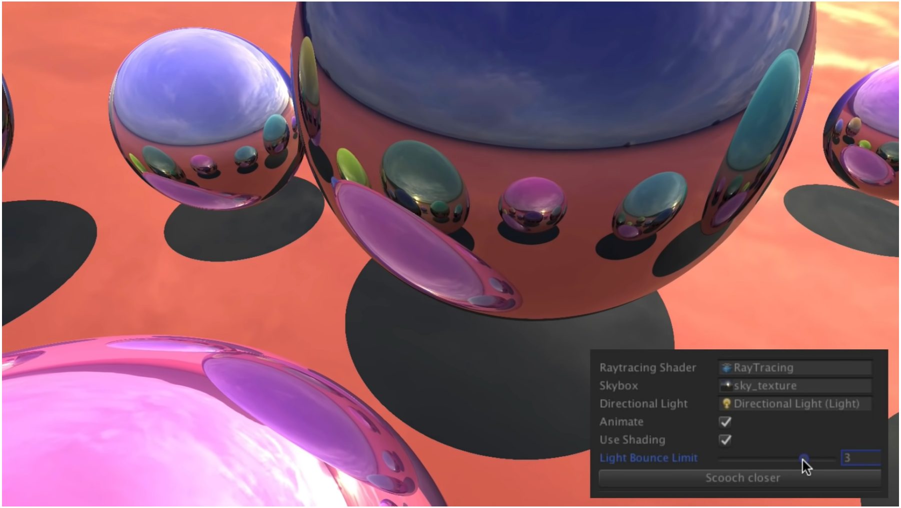

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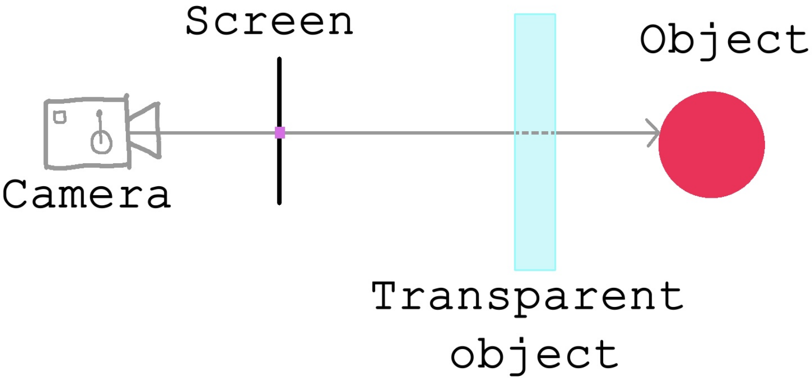

Computer Graphics (Transparency)

Refraction

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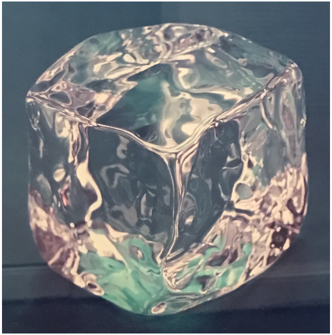

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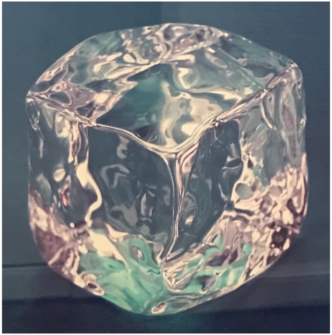

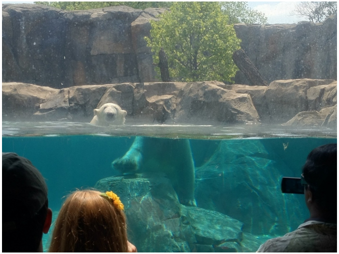

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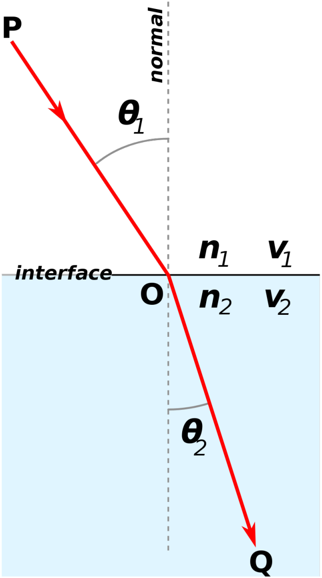

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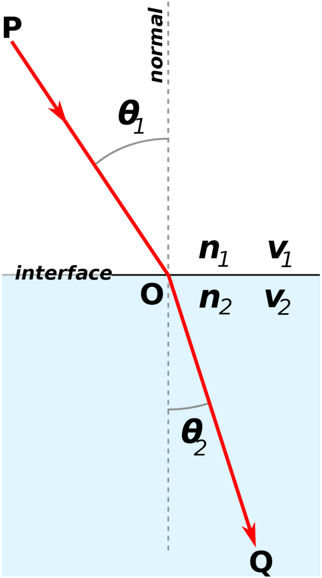

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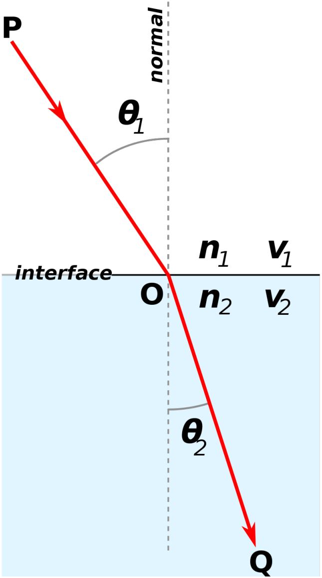

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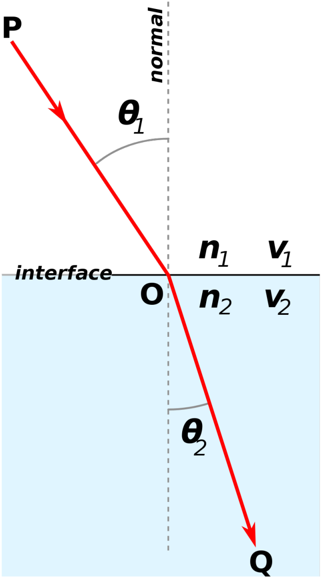

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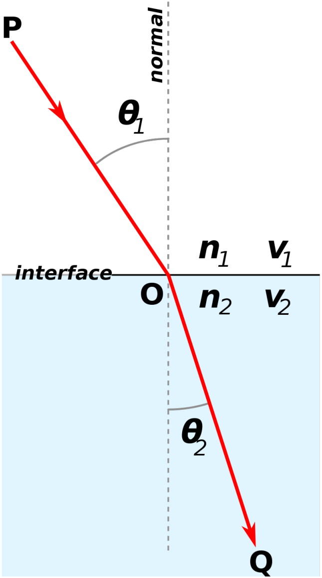

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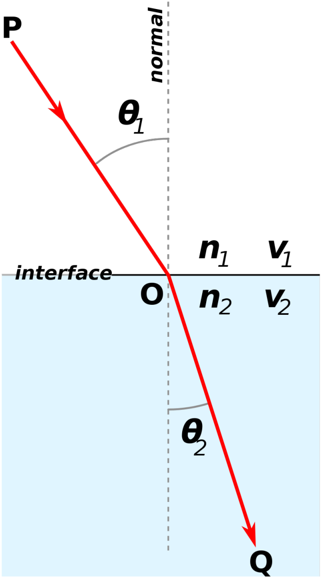

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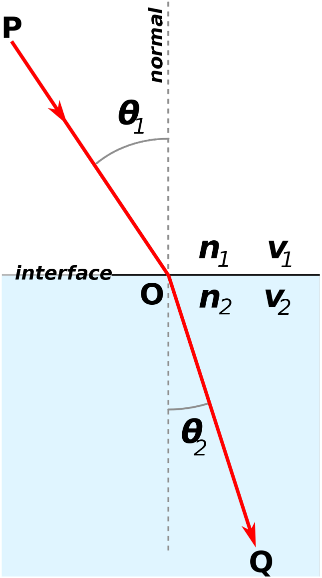

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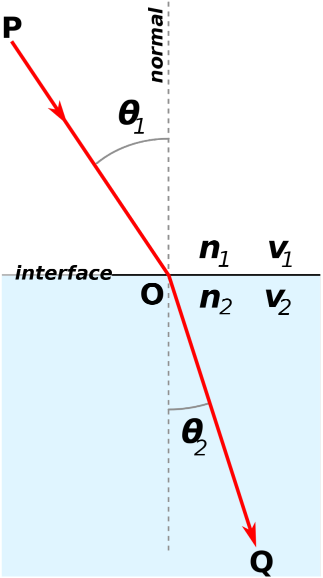

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$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{\mathbf{v}_1}{\mathbf{v}_2} = \frac{\mathbf{n}_2}{\mathbf{n}_1}}$$

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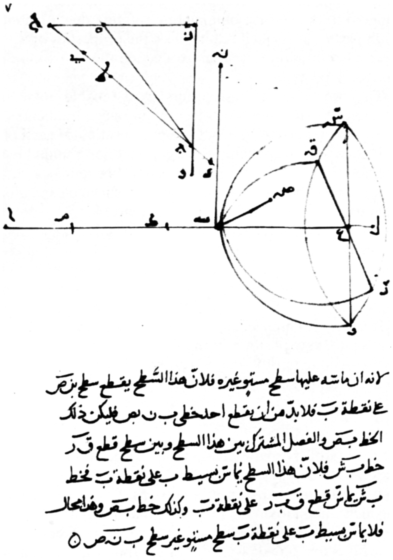

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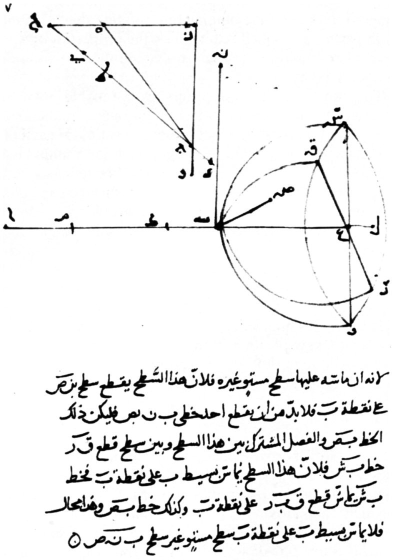

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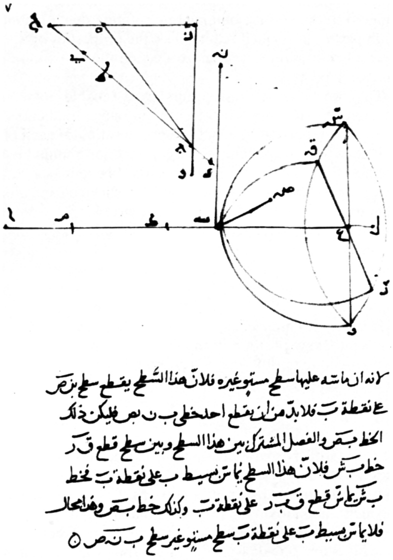

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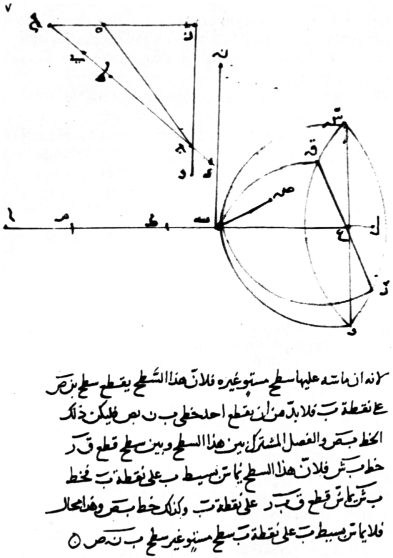

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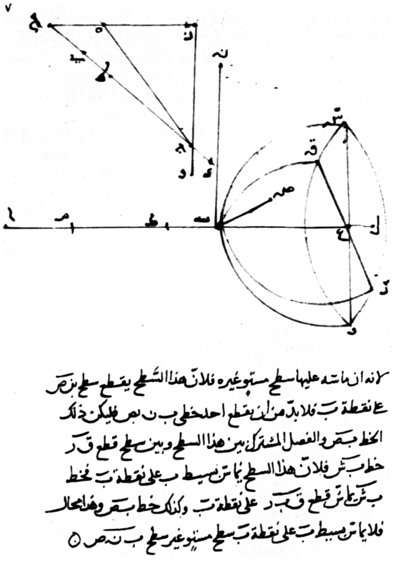
- 1662: Pierre de Fermat rejects Descartes's derivation, reproves the law using his *principle of least time*.



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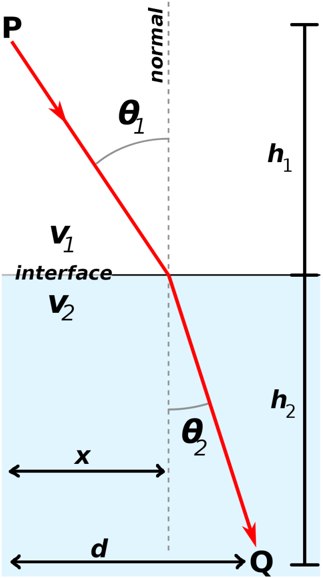

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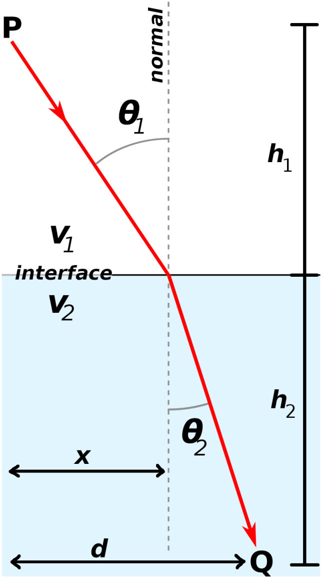

Fermat's Principle: A ray of light follows the path which takes the least time to traverse.

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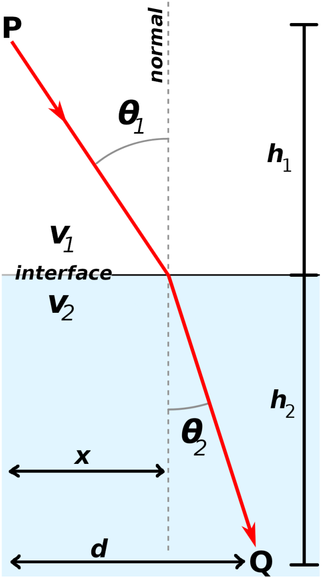

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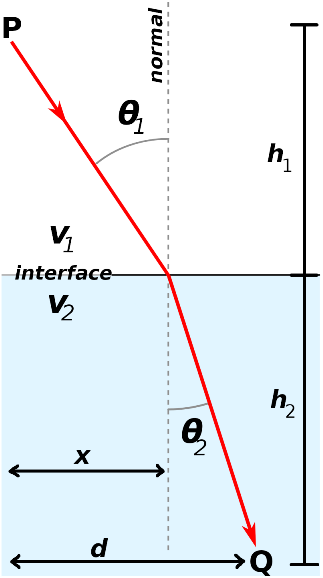

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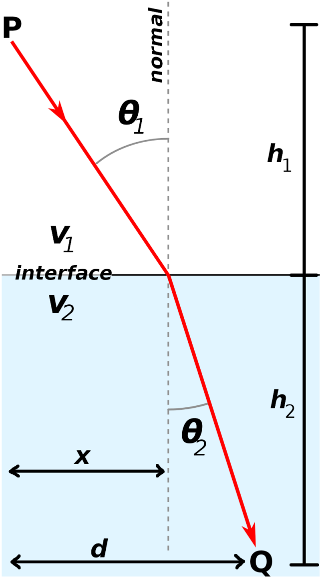

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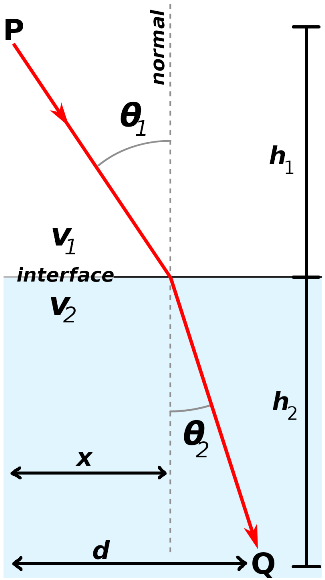

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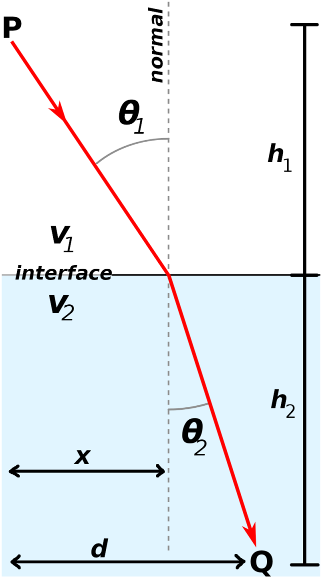

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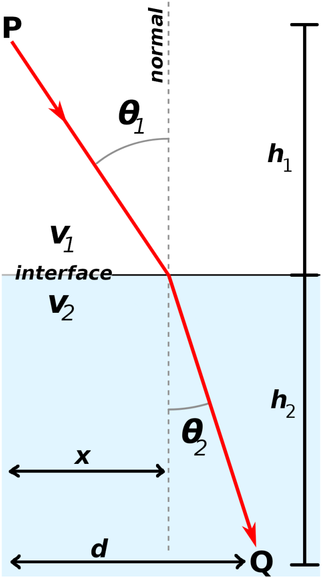

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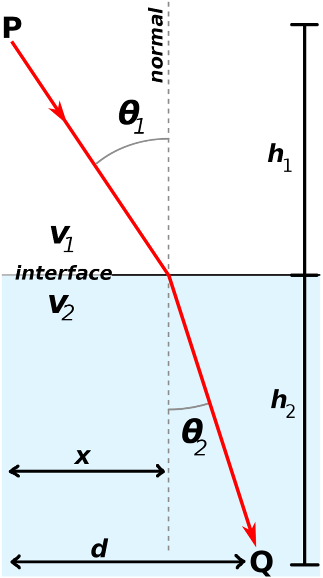

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$$\frac{x}{\sqrt{x^2 + \mathbf{h}_1^2}} = \sin\theta_1, \frac{\mathbf{d} - x}{\sqrt{(\mathbf{d} - x)^2 + \mathbf{h}_2^2}} = \sin\theta_2.$$

Thus $\frac{dT}{dx} = 0 \Leftrightarrow \frac{\sin\theta_1}{\mathbf{v}_1} = \frac{\sin\theta_2}{\mathbf{v}_2} \Leftrightarrow \frac{\sin\theta_1}{\sin\theta_2} = \frac{\mathbf{v}_1}{\mathbf{v}_2}.$

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- Fermat uses this principle to prove the law of refraction, using his own method of *adequality*, and rejecting Descartes's previous proof of the law.

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The Wave Theory of Light

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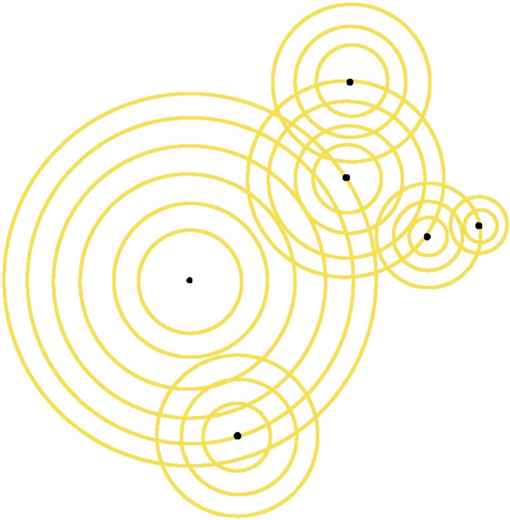

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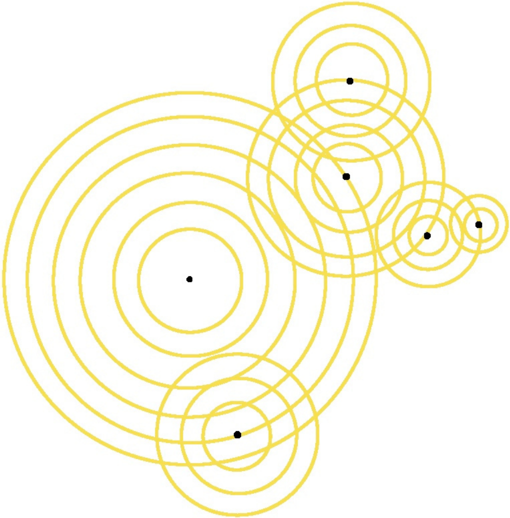

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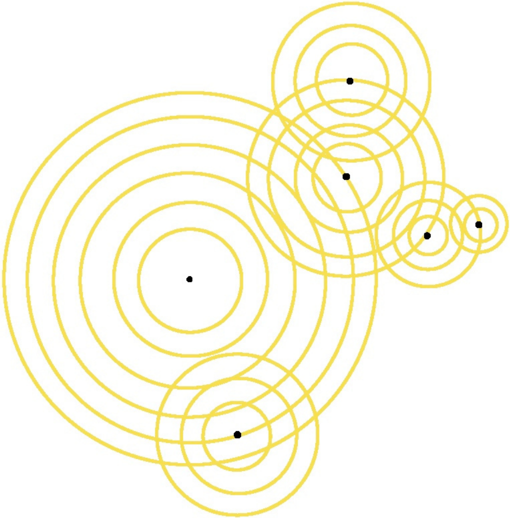

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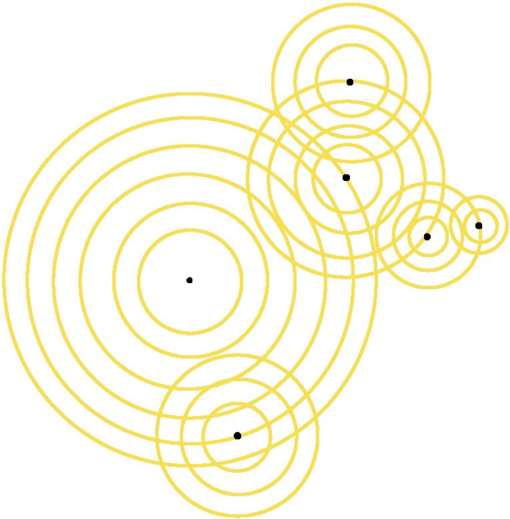

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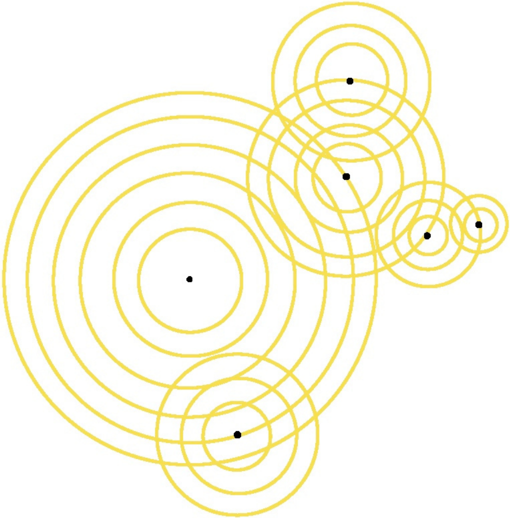

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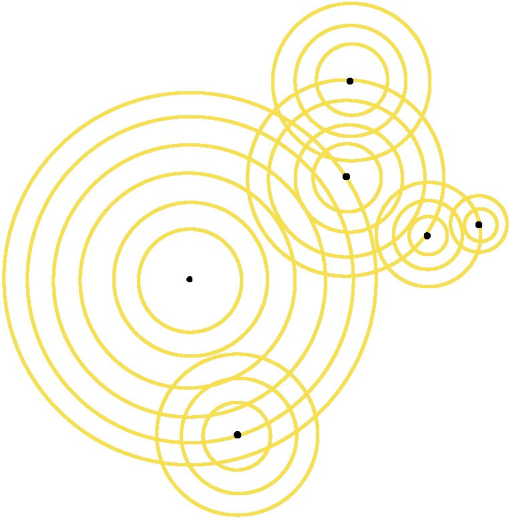

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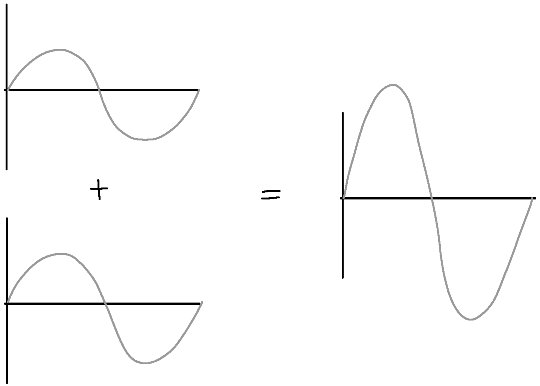

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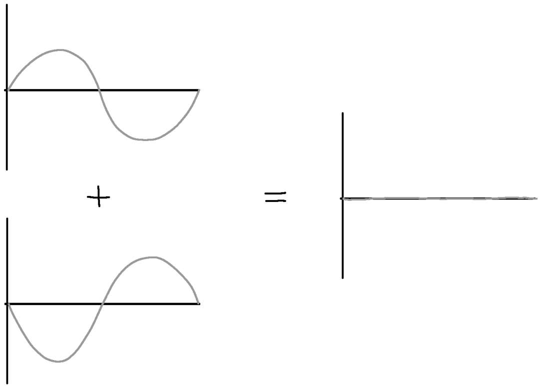

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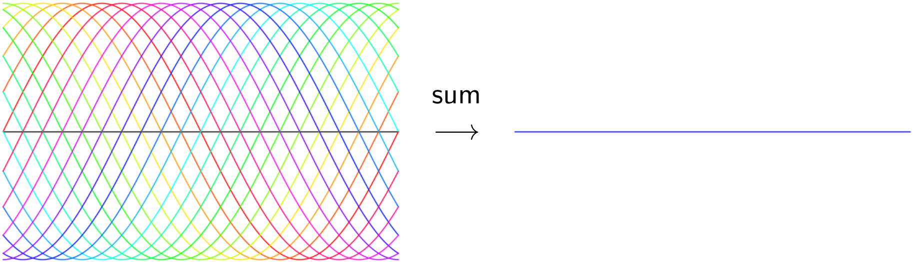

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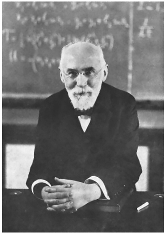

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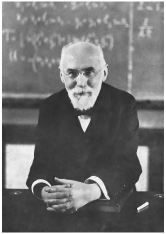
- 1959: Adriaan J. de Witte clarifies Lorentz's ideas and lays out the proof using the *calculus of variations*.



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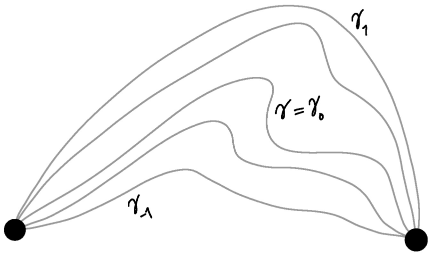

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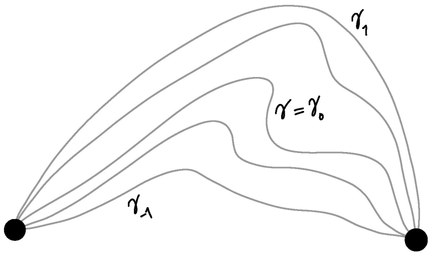

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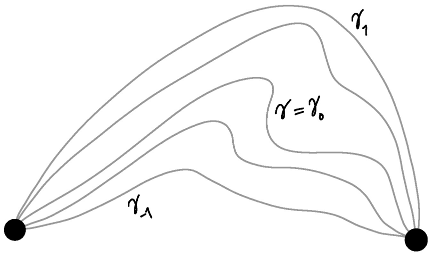

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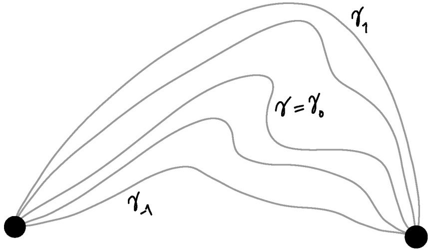

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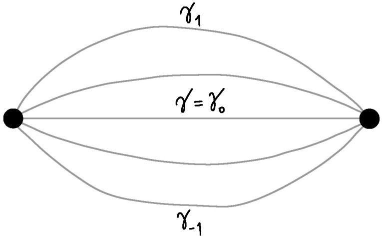

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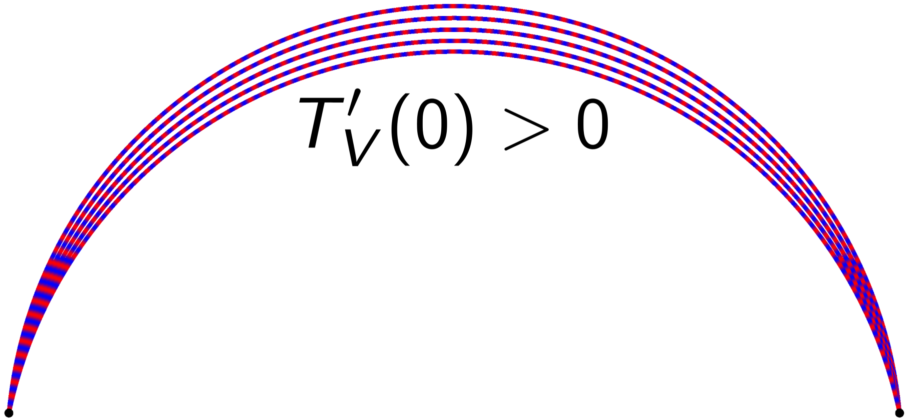

$$T_V'(0) > 0$$

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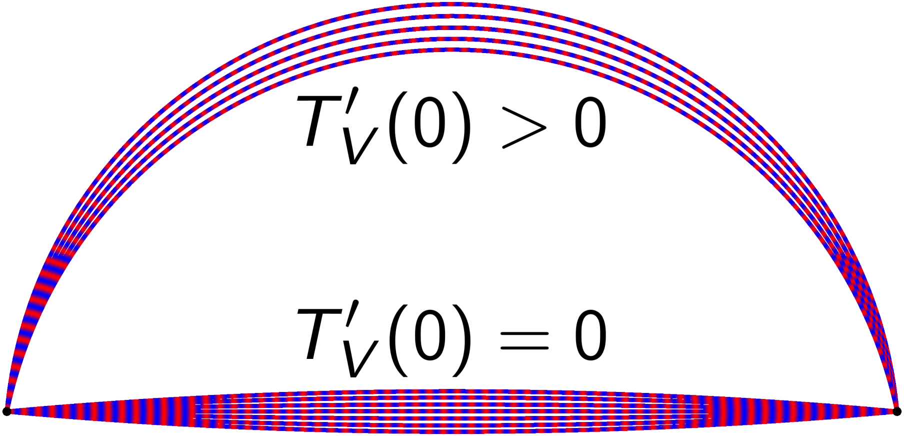

Stationary = Least Time?

Stationary $\neq$ Least Time