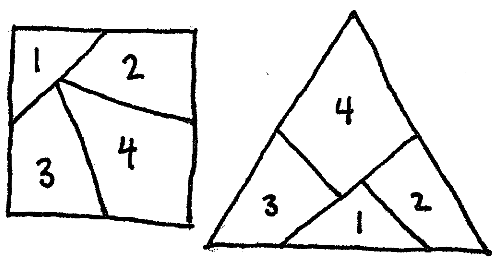
Hilbert's Third Problem

Chapter 0: The Wallace - Bolyai - Gerwien Theorem.

Thm: Two polygons are equidecomposable iff they have same area.

(Equidecomposable means you can cut ~~it~~ one up into finitely many polygonal pieces & rearrange them to form the other):

Eg:



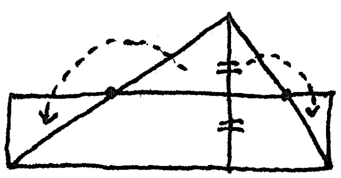
History: Sources vary.

- Some say William Wallace discovered it in 1807,
- Farkas Paul
  others say Bolyai & Gerwien independently discovered it in the early 1830's.

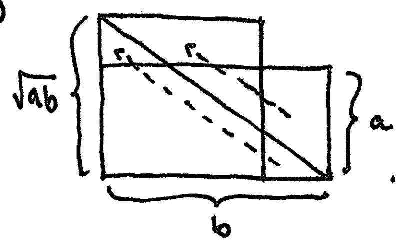
Proof Equidecomposability is an equiv. relation. So we will show that any polygon is equidecomposable with a square of the same area.

① cut up polygon into triangles. For each triangle, do ② and ③

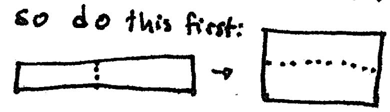
②



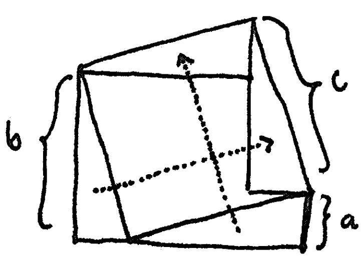
③



③' if $b > 4a$, that picture isn't accurate, so do this first:



④ recombine the resulting squares into one big square by applying the following proof of the Pythagorean thm multiple times.



$(a^2 + b^2 = c^2)$

Done.

Can the analogous theorem hold in three-dimensions?

Chapter 1: History of Hilbert's third Problem.

Gauss & Gerling: In 1844, Gauss responded to a correspondence from Christian Ludwig Gerling (in which Gerling showed that tetrahedra which are mirror images of each other are equidecomposable).

- Gauss expressed dissatisfaction with the fact that the formulae for certain volumes of polyhedra still rely on the method of exhaustion (i.e. calculus & continuity / Cavalieri's Principle).

Hilbert: Inspired by the above correspondence, Hilbert poses his 3rd problem (on his famous 1900 list of 23 problems). He guesses that the method of exhaustion is unavoidable in many polyhedral volume formulae.

- His problem statement is: "Specify two tetrahedra of equal bases and equal altitudes which [are not equidecomposable]."

This problem was certainly not invented by Hilbert. Many people

- In the later 1800's were thinking about equidecomposability of polyhedra. For example, Raoul Bricard Published the following theorem in 1896 (which would lead to a solution to Hilbert's problem).

Thm (Bricard's condition): If two polyhedra are equidecomposable, Then there exist positive integers n_i, m_j, and integer p s.t.

$$n_1\alpha_1 + n_2\alpha_2 + \dots + n_q\alpha_q = m_1\beta_1 + m_2\beta_2 + \dots + m_\ell\beta_\ell + p\pi,$$

where α_i are the measures of dihedral angles of the first polyhedron and β_i are the measures of dihedral angles of the second polyhedron.

unfortunately, Bricard's proof was flawed. Soon after, in 1897, Guiseppe Sforza also published an example of nonequidecomposable polyhedra. Anyway, in 1900, the problem for tetrahedra was still considered open.

Dehn: In 1900 and 1902, Max Dehn (Hilbert's Student) solves the problem! His solution also gives a correct proof of Bricard's Theorem above. ~~In fact~~ Dehn's Original papers are quite complicated, but the arguments therein are eventually simplified by the combined efforts of V.F Kagan (1903/1930), Hugo Hadwiger (1949/1954), and Vladimir Boltianskii (1978).

- An elementary pf of Bricard's condition was found by David Benko in 2007.

- About 60 years after Dehn published his papers introducing Dehn invariants, Jean-Pierre Sydler proved in 1965 that polyhedra of equal volume are equidecomposable if they have the same Dehn invariant.
- In connection to Hilbert's 18th problem (about tiling), Hans Debrunner showed in 1980 that the Dehn invariant of any polyhedron with which 3-D space can be tiled periodically is zero.
- It is still an open problem to determine the analogues to the Dehn invariant in higher dimensions.

The Contest in Kraków: Unbeknownst to Hilbert, his problem had already been solved as early as 1884. Władysław Kretkowski announced a math contest in 1882, with a reward of 500 francs for the solution of almost exactly the same problem as Hilbert posed 18 years later.

- The problem was solved by Ludwik Antoni Birkemajer in a 40 page paper arranged into three chapters. The solution was correct but quite complicated, introducing many invariants with conditional equations. Also, his methods only apply to tetrahedra.
- Birkemajer was a 28-year-old teacher at the time. He was first & foremost a historian of science, publishing many works on the life and work of Nicolaus Copernicus. He also wrote some papers on prime numbers, in particular proving the following:

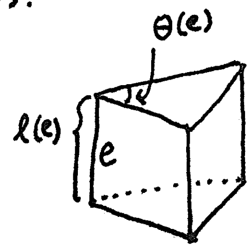
  Thm If the number $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p-1}$ is written as one fraction, its numerator is divisible by p^2 iff p is a prime > 3.

- Birkemajer's solution to Kretkowski's problem went unnoticed and was never published, although it exists somewhere in the scientific library of the Polish Academy of sciences.

Chapter 2: The Dehn Invariant.

Motivation: we'd like some value $D(P)$ (Polyhedron) such that if P is cut into P_1 and P_2, then $D(P) = D(P_1) + D(P_2)$. This would mean, by induction, that if P is cut into $P_1, \dots, P_n$, then $D(P) = D(P_1) + \cdots + D(P_n)$. Thus, if Polyhedra P & Q are equidecomposable, then $D(P) = D(Q)$.

Definition ~~Theorem~~: Let e be an edge of P. Let $\ell(e)$ be the length of e. Let $\theta(e)$ be the dihedral angle at e.



Let $D(P) = \sum_e \ell(e) \otimes (\theta(e) + \mathbb{Q}\pi) \in \underbrace{\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}\pi}_{\text{vector spaces}/\mathbb{Q}}$

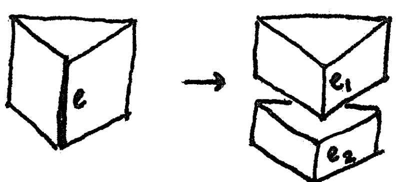
Note: the tensor product notation is not what Dehn originally used, but His construction led to all of the things tensor products do. for a construction closer to Dehn's original, see chapter 8 of "Proofs from the Book" 4^{th} edition (or maybe 5^{th} edition too) not 6^{th}!

Facts about $\otimes$: $\ell \otimes 0 = 0 = 0 \otimes \theta$, or (more generally) $q\ell \otimes \theta = q(\ell \otimes \theta) = \ell \otimes q\theta$. (for $q \in \mathbb{Q}$)
$(\ell_1 + \ell_2) \otimes \theta = \ell_1 \otimes \theta + \ell_2 \otimes \theta$, $\ell \otimes (\theta_1 + \theta_2) = \ell \otimes \theta_1 + \ell \otimes \theta_2$.

Thm: If P is cut into P_1 & P_2, $D(P) = D(P_1) + D(P_2)$.

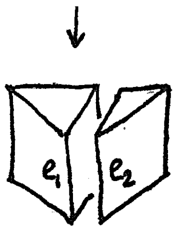
Pf: if an edge e is split by the cut into e_1 and e_2, there are two options:

①



$\ell(e) = \ell(e_1) + \ell(e_2)$, so (since $\theta(e_1) = \theta(e_2) = \theta(e)$),
$\ell(e) \otimes \theta(e) = \ell(e_1) \otimes \theta(e_1) + \ell(e_2) \otimes \theta(e_2)$

②



$\theta(e) = \theta(e_1) + \theta(e_2)$ and $\ell(e_1) = \ell(e_2) = \ell(e)$, so
$\ell(e) \otimes \theta(e) = \ell(e_1) \otimes \theta(e_1) + \ell(e_2) \otimes \theta(e_2)$ in this case too.

New edges created by the splitting are created in pairs $\overset{P_1}{e_1}$ & $\overset{P_2}{e_2}$, and they split up what was a face of P. So $\theta(e_1) + \theta(e_2) = \pi$.
Also, $\ell(e_1) = \ell(e_2) = \ell$, and so $\ell(e_1) \otimes \theta(e_1) + \ell(e_2) \otimes \theta(e_2) = \ell \otimes \pi = \ell \otimes 0 = 0$.

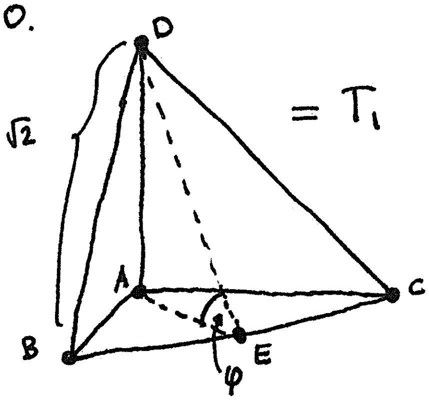
Chapter 3: The solution to Hilbert's third problem.

Tetrahedron 1 is spanned by three orthogonal edges AB, AC, AD, each w/ length 1. Then $\theta(AD)=\theta(AB)=\theta(AC)=\frac{\pi}{2}=0$.

And if φ is $\theta(BC)=\theta(BD)=\theta(CD)$, then

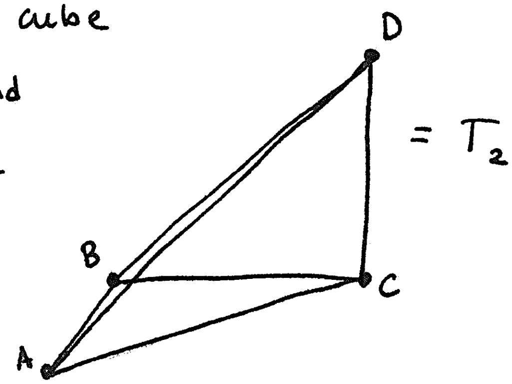
$$\cos\varphi=\frac{|AE|}{|DE|}=\frac{1}{\sqrt{2}}\Big/\frac{\sqrt{3}}{\sqrt{2}}=\frac{1}{\sqrt{3}},$$ and so

$\varphi=\arccos\frac{1}{\sqrt{3}}$, which is not a rational multiple of π. (see PFTB 6e, Chapter 8)



So $D(T_1)=3\left(1\otimes\frac{\pi}{2}\right)+3\left(\sqrt{2}\otimes\arccos\frac{1}{\sqrt{3}}\right)=3\sqrt{2}\otimes\arccos\frac{1}{\sqrt{3}}\neq 0$.

Tetrahedron 2 has three consecutive edges AB, BC, CD which are orthogonal and have length 1. Notice that the unit cube can be decomposed into 3 copies of T_2 and 3 copies of its mirror image (which has the same Dehn invariant). So we have $D(T_2)=\frac{1}{6}D(\text{cube})$. And every edge e in the cube has $\theta(e)=\frac{\pi}{2}=0$, so $D(\text{cube})=0$. Thus $D(T_2)=0$.



T_1 and T_2 have the same bases and altitudes, but they have different Dehn invariants so they are not equidecomposable.

References: Wikipedia and the following:

- Ciesielska & Ciesielski — Equidecomposability of Polyhedra: A solution of Hilbert's Third Problem in Kraków before ICM 1900.
- Tomkowicz & Wagon — The Banach Tarski Paradox (second edition).
- Aigner & Ziegler — Proofs from THE BOOK (fourth edition).
- Numberphile — The Dehn Invariant.