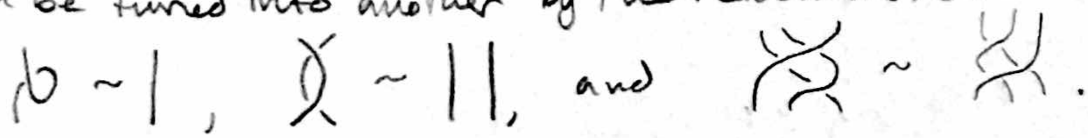
Review of Knots: Knots are continuous injections $S^1 \to \mathbb{R}^3$, question is how to untangle one into another.

two knot diagrams represent the same knot if one can be turned into another by the reidemeister moves:



- A Biquandle is an algebraic object which satisfies some rules similar to these reidemeister moves (for oriented knots).
- It's a set X with two binary operations $\trianglerighteq, \overline{\triangleright}$ satisfying $\forall x,y,z \in X$,

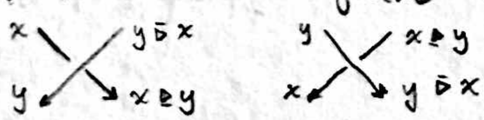
 (i) $x \trianglerighteq x = x \overline{\triangleright} x$

 (ii) The maps $\alpha_y, \beta_y : X \to X$ and $S : X^2 \to X^2$ defined by $\alpha_y(x) = x \overline{\triangleright} y$, $\beta_y(x) = x \trianglerighteq y$, and $S(x,y) = (y \overline{\triangleright} x, x \trianglerighteq y)$ are invertible.

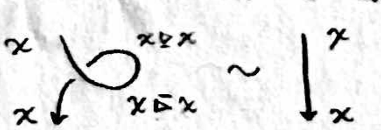
 (iii) the exchange laws are satisfied:

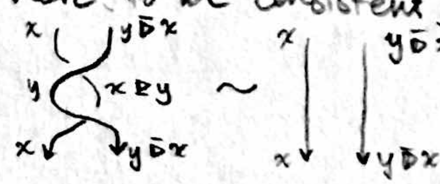
$$(x \trianglerighteq y) \trianglerighteq (z \trianglerighteq y) = (x \trianglerighteq z) \trianglerighteq (y \overline{\triangleright} z)$$
$$(x \trianglerighteq y) \overline{\triangleright} (z \trianglerighteq y) = (x \overline{\triangleright} z) \trianglerighteq (y \overline{\triangleright} z)$$
$$(x \overline{\triangleright} y) \overline{\triangleright} (z \overline{\triangleright} y) = (x \overline{\triangleright} z) \overline{\triangleright} (y \trianglerighteq z)$$

- Where do these rules come from? The motivation is as follows: we interpret the operations as being applied to labeled strands of an oriented knot diagram at a crossing:

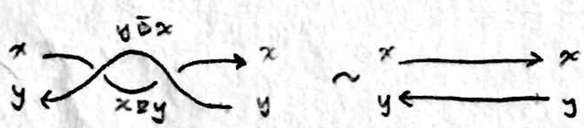


we want the strand labellings here to be consistent so:

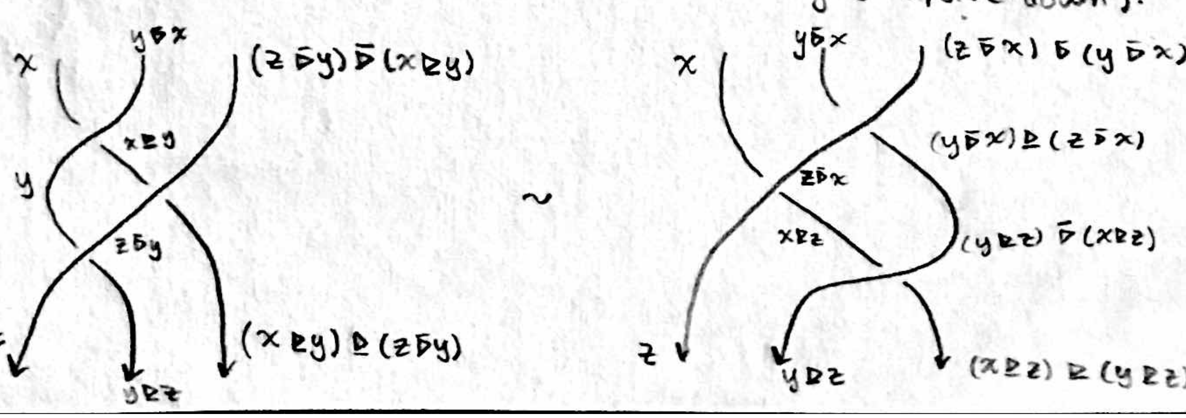
(i)


$(ii)_1$


(strands in same dir.)
this one gives $y \overline{\triangleright} x$ being invertible.

$(ii)_2$


(strands in opposite dir. note the second crossing is upside-down).

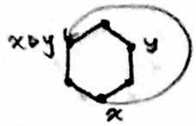
(iii)


Examples of Biquandles:

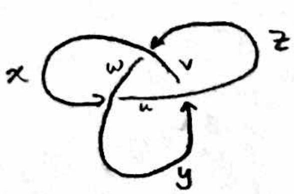
- Let A be a module over $\mathbb{Z}[t^{\pm1}, r^{\pm1}]$. Then A is a biquandle with operations $x \,\underline{\triangleright}\, y = tx + (r^{-1}-t)y$ and $x \,\overline{\triangleright}\, y = r^{-1}y$ called an Alexander Biquandle

- $\mathbb{Z}/_{n\mathbb{Z}}$ is a biquandle with operations $x \,\underline{\triangleright}\, y = 2y - x$, $x \,\overline{\triangleright}\, y = x$.

  if a biquandle's "over" operation is trivial, then it is called a quandle. This is called the Dihedral Quandle / Takasaki kei: (a kei is a quandle where the $\underline{\triangleright}$ operation is an involution).



- Fundamental Biquandle of a Link: generated by arcs in a diagram for L:

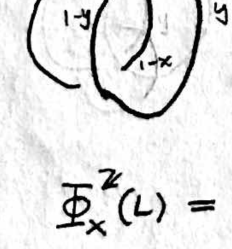
eg: $L =$ 

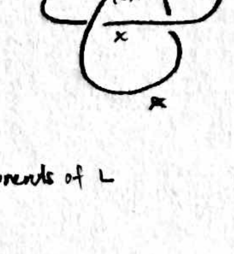
$$B(L) = \left\langle \begin{array}{c} x, y, z \\ u, v, w \end{array} \middle| \begin{array}{l} x \,\underline{\triangleright}\, y = u,\ y \,\overline{\triangleright}\, x = w,\ y \,\underline{\triangleright}\, z = v \\ z \,\overline{\triangleright}\, y = u,\ z \,\underline{\triangleright}\, x = w,\ x \,\overline{\triangleright}\, z = v \end{array} \right\rangle$$

Using the fundamental biquandle we can obtain a few knot invariants: first, the counting invariant:

Pick a biquandle X. Since $B(L)$ is an invariant of L, so is $\mathrm{Hom}(B(L), X)$, and so is $|\mathrm{Hom}(B(L), X)| =: \Phi_X^{\mathbb{Z}}(L) \in \mathbb{N}$.

This can be thought of as the number of X-colorings of any diagram for L. for example, let $X = \{0,1\}$ with $x \,\underline{\triangleright}\, y = x \,\overline{\triangleright}\, y = 1 - x$, a constant-action biquandle.





So $\Phi_X^{\mathbb{Z}}(L) = 2^{\#\text{ of components of } L}$