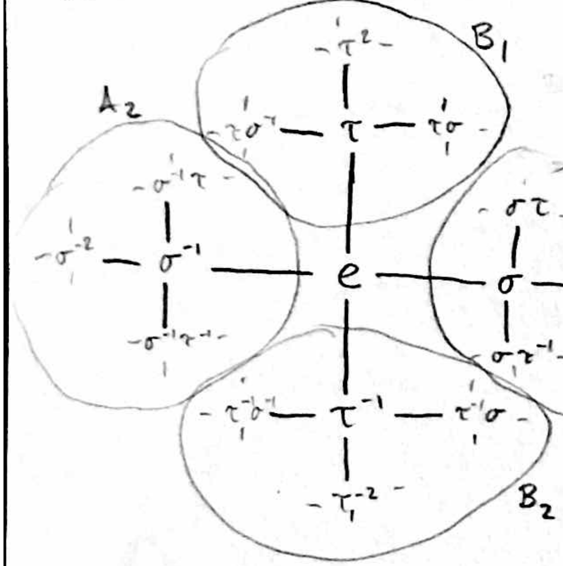
Paradoxical Groups & the Banach-Tarski paradox.

<u>Defn</u>: Let $G \curvearrowright X \supseteq E \neq \emptyset$. E is G-paradoxical if $\exists\, A_1, \ldots, A_n, B_1, \ldots, B_m \subset E$, all pairwise disjoint, And $\exists\, g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ s.t. $E = \bigcup g_i A_i = \bigcup h_j B_j$

<u>Example</u>: $G = X = E = F_2 = \langle \sigma, \tau \rangle$



$eA_1 \cup \sigma A_2 = F_2$

$eB_1 \cup \tau B_2 = F_2$

"F_2 is F_2-paradoxical"

i.e. "F_2 is paradoxical"

<u>Sierpiński-Mazurkiewicz Paradox</u>:

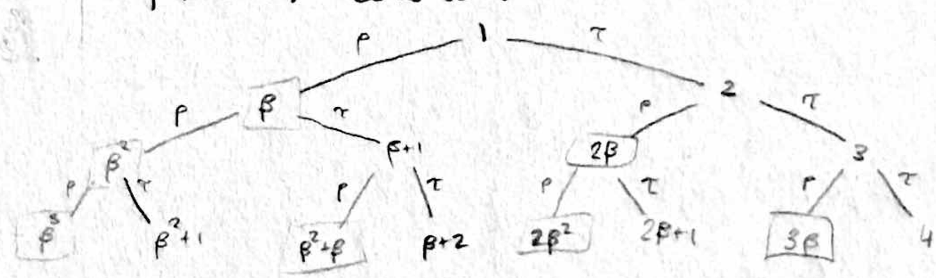
G_2: isometries of $\mathbb{R}^2 = \mathbb{C}$.

$\beta \in \mathbb{C}$ with $|\beta| = 1$.

$\rho : z \longmapsto \beta z$

$\tau : z \longmapsto z + 1$

Look at what $\sigma \in S$ = (semigroup generated by ρ and τ) does to 1:



thus different "words" in ρ and τ yield different polynomials in the "variable" β.

So if we pick β to be transcendental, then $\langle \rho, \tau \rangle \leq G_2$ is free: if σ_1 and σ_2 are two distinct words in ρ, τ and $P_1(\beta) = \sigma_1(1)$, $P_2(\beta) = \sigma_2(1)$, then $P_1(x) \neq P_2(x)$ but $P_1(\beta) - P_2(\beta) = 0$ so β is the root of a polynomial.

So take A = boxed elts, B = unboxed elts (other than 1), and let $g = \rho^{-1}$, $h = \tau^{-1}$. Thus $E = \{\sigma(1) : \sigma \in S\}$ is G_2-paradoxical.

Theorem: If G is paradoxical & $G \curvearrowright X$ w/ no nontrivial fixed points, then X is G-paradoxical.

Requires AC.

Proof: Let M be a set of representatives of the G-orbits of X. Then $\{g(M) : g \in G\}$ is a partition of X. [By definition, $\bigcup_{g \in G} g(M) = X$, and if $g_1(M) \cap g_2(M) \ni x$ then $g_1 y = g_2 z = x$ but then $g_1^{-1}x = y, g_2^{-1}x = z$ so y & z are in the same orbit so $y = z$. Thus $g_1^{-1}g_2 y = y \Rightarrow g_1 = g_2$.]

If $S \subseteq G$, let $S^* = \{g(M) : g \in S\} \subseteq X$. Let $A_i, B_j \subset G$ and $g_i, h_j \in G$ bear witness to G's paradoxicality. Then $X = \bigcup g_i A_i^* = \bigcup h_j B_j^*$, and A_i^*, B_j^* are pairwise disjoint so X is G-paradoxical.

Example: take ϕ and ρ to be rotations of $180°$ and $120°$ about axes which meet at an angle of θ where $\cos 2\theta$ is transcendental. Then $\langle \phi, \rho \rangle = F_2$.

Hausdorff Paradox: There is a countable subset $D \subseteq S^2$ s.t. $S^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical.

$\rightarrow$ Take $D = \bigcup_{\sigma \in \langle \phi, \rho \rangle}$ fixed points in S^2 of σ. each σ has 2 fixed points (it is a nontrivial rotation about an axis L, and $L \cap S^2$ are the fixed points).

Then $\langle \phi, \rho \rangle \curvearrowright S^2 \setminus D$ with no nontrivial fixed points.

For the full banach-tarski paradox (S^2 is $SO_3(\mathbb{R})$-paradoxical) we need some more machinery.

Defn: Suppose $G \curvearrowright X \supseteq A, B$. $A \sim_G B$ if $A = \bigsqcup_{i=1}^n A_i$, $B = \bigsqcup_{i=1}^n B_i$ and $\exists\, g_1, \dots, g_n \in G$ s.t. $g_i A_i = B_i$ $\forall i$. (G-equidecomposable)

So: E is G-paradoxical iff $\exists\, A, B \subseteq E$ s.t. $A \cap B = \emptyset$ and $A \sim_G E \sim_G B$.

Proposition: Suppose $G \curvearrowright X \supseteq E, E'$ with $E \sim_G E'$. Then if E is G-equidecomposable, so is E'.

Proof: $\exists\, A, B \subseteq E$ with $A \cap B = \emptyset$, s.t. $A \sim E \sim B$. Then the equidecomposition $E \sim E'$ gives $A \sim A'$ and $B \sim B'$ for subsets $A', B' \subseteq E'$ with $A' \cap B' = \emptyset$.
And $A' \sim A \sim E \sim E' \sim E \sim B \sim B'$.

Theorem: If $D \subseteq S^2$ is countable, $S^2 \sim_{SO_3(\mathbb{R})} S^2 \setminus D$.

Trick: Absorption of small things to render them irrelevant.

proof: We seek a rotation ρ s.t. $D, \rho(D), \rho^2(D), \ldots$ are all pairwise Disjoint. Then if $D^* = \bigcup_{n\geq 0} \rho^n(D)$, we have $D^* \sim \rho(D^*) = \bigcup_{n\geq 1} \rho^n(D)$ so

$$S^2 = D^* \cup (S^2 \setminus D^*) \sim \rho(D^*) \cup (S^2 \setminus D^*) = S^2 \setminus D.$$

To find this ρ, we let ℓ be a line through the origin which misses the countable set D.

Let $A = \{\theta : \exists\, n>0,\ x \in D$ s.t. $\rho_\theta^n(x) \in D$, where ρ_θ is rotation by θ thru $\ell\}$.

$A = \bigcup_{x\in D} \left(\bigcup_{n>0} \{\theta : \rho_\theta^n(x) \in D\} \right)$ is countable as a union of countable sets, so $\exists\ \theta_0 \notin A$. take $\rho = \rho_{\theta_0}$.

Banach-Tarski Paradox: S^2 is $SO_3(\mathbb{R})$-paradoxical. Also, Any thick Spherical shell is G_3-paradoxical. And, any solid ball is G_3-paradoxical.

for simplicity, we'll show $B = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ is G_3-paradoxical. First, $B \setminus \{0\}$ is by the above reasoning. Then $B \sim B \setminus \{0\}$ as follows:

Defn: we say $A \preccurlyeq B$ if $A \sim B_1$ for some subset $B_1 \subseteq B$.

Theorem (Banach-Schröder-Bernstein): If $A \preccurlyeq B$ and $B \preccurlyeq A$ then $A \sim B$.

Proof: Note that if $A \sim B$ then $\exists$ a bijection $f: A \to B$ s.t. $C \sim f(C)$ whenever $C \subseteq A$. $(*)$

Let $f: A \to B_1$, $g: A_1 \to B$ be such bijections for $A_1 \subseteq A$, $B_1 \subseteq B$.

Let $C_0 = A \setminus A_1$, $C_{n+1} = g^{-1}(f(C_n))$. Let $C = \bigcup_{n=0}^{\infty} C_n$. Then $g(A \setminus C) = g(A_1 \setminus C) = B \setminus \bigcup_{n\geq 1} g(C_n) = \underbrace{B \setminus \bigcup_{n\geq 0} f(C_n)}_{= B \setminus f(C)}$

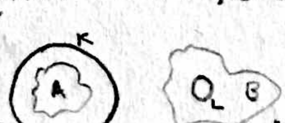
note $A_1 \cap C_0 = \emptyset$.

So $A \setminus C \sim B \setminus f(C)$ by $(*)$.

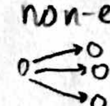
also $C \sim f(C)$ by $(*)$. So $A = (A \setminus C) \cup C \sim (B \setminus f(C)) \cup f(C) = B$.

Banach-Tarski Paradox (strong form): If $A, B \subseteq \mathbb{R}^3$ are bounded & have non-empty interior, then $A \sim_{G_3} B$.

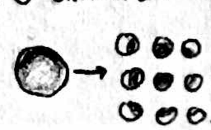
Pf:



$K \subseteq \bigcup^{\text{finite}}$ shift of L:



So



S Using BTP, $L \succcurlyeq S$.

So $A \subseteq K \preccurlyeq S \preccurlyeq L \subseteq B$ so $A \preccurlyeq B$.