Paradoxical Group Actions and the Banach-Tarski Paradox

Definition: Let G be a group acting on a set X, and let $E \subseteq X$ be nonempty. E is *G-paradoxical* if there are pairwise disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq E$ and elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that $E = \bigcup_{i=1}^n g_i A_i = \bigcup_{j=1}^m h_j B_j$.

Example: Let $G = X = E = F_2 = \langle \sigma, \tau \rangle$, where the action is left multiplication. Let $W(x)$ be the set of words which begin with x when fully reduced. Let $A_1 = W(\sigma)$, $A_2 = W(\sigma^{-1})$, $B_1 = W(\tau)$, and $B_2 = W(\tau^{-1})$. Let $g_1 = h_1 = e$, and let $g_2 = \sigma, h_2 = \tau$. Thus F_2 is F_2-paradoxical. To shorten notation, F_2 is paradoxical.

Sierpiński-Mazurkiewicz Paradox: Let $G = G_2$ be the group of isometries of $\mathbb{R}^2 = \mathbb{C}$, and let $X = \mathbb{C}$. Let $\beta \in \mathbb{C}$ be a transcendental number satisfying $|\beta| = 1$. Let $\rho : z \mapsto \beta z$, and let $\tau : z \mapsto z + 1$. Then the subsemigroup S of G_2 generated by ρ and τ is isomorphic to the free semigroup Let $E = \{\sigma(1) : \sigma \in S\}$. Then E is G_2-paradoxical with $A = \beta E$, $B = E + 1$, $g = \rho^{-1}$ and $h = \tau^{-1}$.

Theorem: If G is paradoxical and acts on X without nontrivial fixed points, then X is G-paradoxical. Hence X is F-paradoxical whenever $F \cong F_2$ acts on X with no nontrivial fixed points. Note: this requires the axiom of choice.

Example: The first example of a free subgroup of $SO_3(\mathbb{R})$ was given by Hausdorff: take ϕ and ρ to be rotations of π and $\frac{2\pi}{3}$ about axes which meet at an angle of θ where $\cos(2\theta)$ is transcendental. Then $\langle \rho\phi\rho, \phi\rho\phi\rho\phi \rangle \cong F_2$. Another (more explicit) example, due to Satô, is $SO_3(\mathbb{R}) \geq \langle \sigma, \tau \rangle \cong F_2$ where σ and τ are the *Satô rotations*:

$$\sigma = \frac{1}{7}\begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{bmatrix} \qquad \text{and} \qquad \tau = \frac{1}{7}\begin{bmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{bmatrix}.$$

Hausdorff Paradox: There is a countable subset D of $\mathbb{S}^2$ such that $\mathbb{S}^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical.

Definition: Suppose G acts on X and $A, B \subseteq X$. A and B are said to be *G-equidecomposable* (denoted $A \sim_G B$) if $A = \bigcup_{i=1}^n A_i$ and $B = \bigcup_{i=1}^n B_n$ so that $A_i \cap A_j = \varnothing = B_i \cap B_j$ if $i \neq j$ and there are $g_1, \ldots, g_n \in G$ so that $g_i(A_i) = B_i$ for all i. Note: $\sim_G$ is an equivalence relation. Also, if A is G-equidecomposable with a subset of B, we write $A \preccurlyeq_G B$.

Proposition: Suppose G acts on X, and $E, E' \subseteq X$ with $E \sim_G E'$. Then E is G-paradoxical if, and only if, E' is G-paradoxical.

Theorem: If D is a countable subset of $\mathbb{S}^2$, then $\mathbb{S}^2$ and $\mathbb{S}^2 \setminus D$ are $SO_3(\mathbb{R})$-equidecomposable.

Banach-Tarski Paradox: Any closed ball $B = \{x \in \mathbb{R}^3 : |x - c| \leq r\}$ is G_3-paradoxical (where G_3 is the group of isometries of $\mathbb{R}^3$).

Theorem (Banach-Schröder-Bernstein): Suppose G acts on X, and $A, B \subseteq X$. If $A \preccurlyeq_G B$ and $B \preccurlyeq_G A$ then $A \sim_G B$.

Banach-Tarski Paradox +1: If A and B are any two bounded subsets of $\mathbb{R}^3$, each having non-empty interior, then $A \sim_{G_3} B$ (where G_3 is the group of isometries of $\mathbb{R}^3$).

Application: If G acts on X and X contains a G-paradoxical subset, then there cannot exist a finitely additive G-invariant probability measure defined on all of $\mathscr{P}(X)$. Thus there is no such measure for $X = \mathbb{S}^2$ and $G = SO_3(\mathbb{R})$, or for $X = \mathbb{R}^3$ and $G = G_3$.

Exercise: Show that G_1 and G_2 are solvable groups, and use this fact (or something else) to show that neither G_1 nor G_2 contain a free subgroup. Thus there is no analogous construction in $\mathbb{R}^2$ (or $\mathbb{R}$) to the Banach-Tarski paradox in $\mathbb{R}^3$. Note that this doesn't mean there are no G_2-paradoxical subsets of $\mathbb{R}^2$: the Sierpiński-Mazurkiewicz paradox above gives one. However, the group generated by $\tau, \rho \in G_2$ given in that example do not generate a free group since $\tau\rho\tau^{-1}\rho^{-1}$ commutes with $\tau^{-1}\rho^{-1}\tau\rho$ (chek this) and this gives a nontrivial relation in $\langle \tau, \rho \rangle$.

Reference: Grzegorz Tomkowicz and Stan Wagon. *The Banach-Tarski Paradox.* Cambridge University Press, 2016.