Pigeonhole Principle:

Useful statement: suppose N and R are two sets and $\underset{n}{|N|} > \underset{r}{|R|}$. Let $f: N \to R$.

then there is some $a \in R$ with $|f^{-1}(a)| \geq 2$.

Stronger: there is $a \in R$ with $|f^{-1}(a)| \geq \lceil \frac{n}{r} \rceil$. if not, $|f^{-1}(a)| < \frac{n}{r}$ for all $a \in R$ and so $n = \sum_{a \in R} |f^{-1}(a)| < r \frac{n}{r} < n$.

Question: take any $n+1$ numbers from $\{1, 2, 3, \dots, 2n\}$. are there two coprime?

A: yes, there must be two which are 1 apart, and these are coprime.

Question: do the same thing, are there two s.t. one divides the other?

A: yes, let each $k = 2^\ell m^{\text{odd}}$. there are at most n choices for m.

Q: Are these results true if you replace $n+1$ by n?

A: no. $\{2, 4, \dots, 2n\}$ and $\{n+1, n+2, \dots, 2n\}$.

Theorem 1. Let $a_1, \dots, a_n \in \mathbb{Z}$. for some k, ℓ with $0 \leq k < \ell \leq n$, we have $\sum_{i=k+1}^{\ell} a_i \equiv 0 \pmod{n}$.

Proof Let $f(m)$ be the remainder of $a_1 + \cdots + a_m$ upon division by n. There are n possible values of $f(m)$, and if some $f(m) = 0$ we are done. otherwise, f maps $\{1, \dots, n\}$ to $\{1, \dots, n-1\}$ so we must have $f(m_1) = f(m_2)$ for some $m_1 < m_2$ (Pigeonhole principle). Thus

$$\sum_{i=m_1+1}^{m_2} a_i = \sum_{i=1}^{m_2} a_i - \sum_{i=1}^{m_1} a_i \equiv f(m_2) - f(m_1) = 0 \pmod{n}.$$

Theorem 2: Let $(a_i)_{i=1}^{mn+1}$ be a finite sequence of $mn+1$ distinct real numbers. Then there is an increasing subsequence

$$a_{i_1} < a_{i_2} < \cdots < a_{i_{m+1}} \qquad (i_1 < i_2 < \cdots < i_{m+1})$$

of length $m+1$, or a decreasing subsequence

$$a_{j_1} > a_{j_2} > \cdots > a_{j_{n+1}} \qquad (j_1 < j_2 < \cdots < j_{n+1})$$

of length $n+1$, or both.

Proof Let t_i be the length of the longest increasing subsequence starting at a_i. If any $t_i \geq m+1$, we are done. Otherwise, $i \longmapsto t_i$ maps $\{1, \ldots, mn+1\}$ to $\{1, \ldots, m\}$, so for some $s \in \{1, \ldots, m\}$ we have $t_j = s$ for $\lceil \frac{mn+1}{m} \rceil = n+1$ numbers $j \in \{1, \ldots, mn+1\}$. (Strong Pigeonhole principle).
Let $j_1 < \cdots < j_{n+1}$ be these numbers. If $a_{j_i} < a_{j_{i+1}}$ then we would have an increasing sequence of length $s+1$ starting at a_{j_i}, which contradicts $t_{j_i} = s$. So

$$a_{j_1} > a_{j_2} > \cdots > a_{j_{n+1}}$$

and we are done.

Double-Counting:

Statement. Let R and C be finite sets, $S \subseteq R \times C$. Then

$$\sum_{p \in R} \left|\{q \in C : (p,q) \in S\}\right| = |S| = \sum_{q \in C} \left|\{p \in R : (p,q) \in S\}\right|.$$

Theorem 3: Let $\tau(j)$ denote the number of divisors of j. Let $\overline{\tau}(n)$ be the average value of $\tau(j)$ for $j \leq n$: $\overline{\tau}(n) = \frac{1}{n} \sum_{j=1}^{n} \tau(j)$.
Then $\overline{\tau}(n) \sim \log n$ (in fact these two quantities differ by less than 1).

Proof: Observe the chart below where we place a dot if $r \mid c$:

$r \backslash c$	1	2	3	4	5	6	7	8
1	•	•	•	•	•	•	•	•
2		•		•		•		•
3			•			•		
4				•				•
5					•			
6						•		
7							•	
8								•

The number of dots is $\sum_{j=1}^{n} \tau(j)$ (counting column-first). But the number of dots in row i is $\lfloor \frac{n}{i} \rfloor$, so we have

$$\overline{\tau}(n) = \frac{1}{n} \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor \leq \frac{1}{n} \sum_{i=1}^{n} \frac{n}{i} = \sum_{i=1}^{n} \frac{1}{i} \sim \log n.$$

but the error when going from $\lfloor \frac{n}{i} \rfloor$ to $\frac{n}{i}$ is less than 1.
So we also have $\overline{\tau}(n) > \sum_{i=1}^{n} \frac{1}{i} - 1 \sim \log n$.

Unrelated question (important!): Let $G = (V, E)$ be a graph, and $d(v)$ be the degree of $v \in V$. Show that $\sum_{v \in V} d(v) = 2|E|$.

Answer: each edge is incident on two vertices, so we'll see each edge twice in the sum.

Theorem 4. The number of trees on n vertices is $T_n = n^{n-2}$.

Proof: We will count* the number of sequences of directed edges that can be added to an empty graph on n vertices in order to form a rooted & directed tree.

* in two ways of course!

Way 1: Start with any undirected tree, choose one of its n vertices to be the root, and then add the $n-1$ edges as directed edges in any order.

This gives $T_n \cdot n \cdot (n-1)! = T_n \cdot n!$.

Way 2: Add directed edges one-by-one to an empty graph.

Count the number of choices at each step:

If we have already added $n-k$ edges then we will have a forest of k rooted & directed trees.

The edge we add at this step can start at any of the n vertices and end at the root of any of the $k-1$ trees which don't contain the starting vertex. So the number of ways is

$$\prod_{k=2}^{n} n(k-1) = n^{n-1}(n-1)! = n^{n-2} \cdot n!$$

<u>Sperner's Lemma</u>: Suppose some "big" triangle with vertices V_1, V_2, V_3 is triangulated. Color the vertices in the triangulation by the colors c_1, c_2, c_3 so that V_i gets color c_i, and only the colors c_i and c_j are used for the vertices between V_i and V_j (the vertices inside the big triangle can be colored arbitrarily). Then there is a small tricolored triangle in the triangulation.

<u>Proof</u>: Consider the subgraph of the dual graph of the triangulation obtained by taking all edges which cross over an edge of the triangulation which has colors c_1 and c_2 (one of each).

the vertices of this partial dual graph have degree

1   if they are in a tricolored triangle

0 or 2   if they are not in a tricolored triangle
(2 if the triangle has both c_1 and c_2, 0 if not)

and the vertex outside the triangle has odd degree since there must be an odd number of color changes between V_1 (which has color c_1) and V_2 (which has color c_2).

thus, for $\sum$ degrees to be even, there must be an odd number of tricolored triangles.

(Note: the triangulation doesn't have to be as regular as it is on the handout)

Brouwer's Fixed Point Theorem $(n=2)$: $f: B_2 \xrightarrow{cts} B_2$ has a fixed point.

Proof: ◦ Let Δ be the triangle in $\mathbb{R}^3$ with vertices $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$. It suffices to prove $f: \Delta \xrightarrow{cts} \Delta$ has a fixed point since Δ is homeomorphic to B_2.

◦ Let $\delta(T)$ denote the maximal length of an edge in a triangulation T of Δ. We can easily construct a sequence of triangulations $T_1, T_2, \ldots$ of Δ so that $\delta(T_k) \longrightarrow 0$ as $k \longrightarrow \infty$.

◦ for each triangulation, define a 3-coloring of the vertices v by coloring v with $c_{\lambda(v)}$ where $\lambda(v) := \min\{i : f(v)_i < v_i\}$; $\lambda(v)$ is the smallest index i so that the i^{th} coordinate of $f(v) - v$ is negative. If this smallest i does not exist, we have found v so that $f(v) = v$. to see this, note that every $u \in \Delta$ satisfies $u_1 + u_2 + u_3 = 1$, so if $f(v) \neq v$ then at least one coordinate of $f(v) - v$ must be negative (and one must be positive).

◦ We check that this coloring satisfies the conditions of Sperner's lemma: e_i must receive color c_i since the only possible negative component of $f(e_i) - e_i$ is the i^{th} component.
Also, if v is in the edge of Δ opposite to e_i, then $v_i = 0$, so the i^{th} component of $f(v) - v$ cannot be negative. (v doesn't get c_i).

◦ Now Sperner's lemma says in each triangulation T_k there is a tricolored triangle $\{v^{k,1}, v^{k,2}, v^{k,3}\}$ with $\lambda(v^{k,i}) = i$.
$(v^{k,1})_{k=1}^{\infty}$ may not converge, but a subsequence does since Δ is compact.
Assume we started with this subsequence so $(v^{k,1})_{k=1}^{\infty}$ does converge to a point $v \in \Delta$. Now $(v^{k,2})$ and $(v^{k,3})$ also converge to v since $|v^{k,2} - v^{k,1}| \leq \delta(T_k) \geq |v^{k,3} - v^{k,1}|$ and $\delta(T_k) \longrightarrow 0$.

◦ We know $f(v^{k,1})_1 < v_1^{k,1}$ for all k (by definition of λ), so, since f is continuous, $f(v)_1 \leq v_1$. but the same reasoning gives $f(v)_2 \leq v_2$ and $f(v)_3 \leq v_3$.
So no coordinate of $f(v) - v$ is positive, which means we cannot have $f(v) \neq v$.