Pigeon-Holes and Double-Counting

Math 5529H - September 19, 2018

Some scattered results

Theorem 1. *Suppose we are given n integers $a_1, \ldots, a_n$, which need not be distinct. Then there is always a set of consecutive numbers $a_{k+1}, a_{k+2}, \ldots, a_\ell$ whose sum $\sum_{i=k+1}^{\ell} a_i$ is a multiple of n.*

Theorem 2. *In any sequence $a_1, a_2, \ldots, a_{mn+1}$ distinct real numbers, there exists an increasing subsequence*

$$a_{i_1} < a_{i_2} < \cdots < a_{i_{m+1}} \qquad (i_1 < i_2 < \cdots < i_{m+1})$$

of length $m+1$, or a decreasing subsequence

$$a_{j_1} > a_{j_2} > \cdots > a_{j_{n+1}} \qquad (j_1 < j_2 < \cdots < j_{n+1})$$

of length $n+1$, or both.

Theorem 3. *Let $\tau(j)$ denote the number of divisors of j. Let $\overline{\tau}(n)$ denote the average value of $\tau(j)$ for $j \leq n$: $\overline{\tau}(n) = \frac{1}{n}\sum_{j=1}^{n} \tau(j)$. Then $\overline{\tau}(n) \sim \log n$. In fact, these two quantities differ by less than 1.*

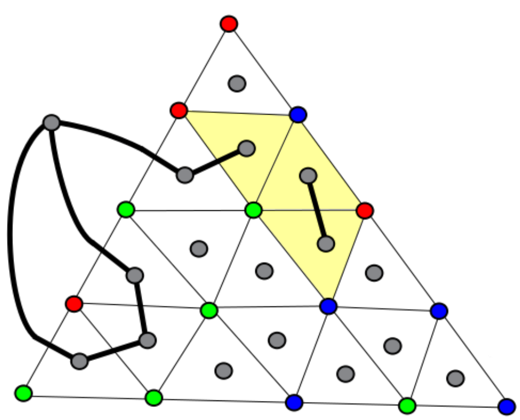
Theorem 4. *The number of trees on n vertices is $T_n = n^{n-2}$.*

Sperner's Lemma and Brouwer's Fixed-Point theorem

Sperner's Lemma. *Suppose that some "big" triangle with vertices V_1, V_2, V_3 is triangulated (that is, decomposed into a finite number of "small" triangles that fit together edge-by-edge).*

Assume that the vertices in the triangulation get colored with three colors c_1, c_2, c_3 so that V_i is colored c_i and only the colors c_i and c_j are used for vertices along the edge (of the "big" triangle) from V_i to V_j (here $i \neq j$), while the interior vertices are colored arbitrarily.

Then in the triangulation there must be a small "tricolored" triangle, which has all three different vertex colors. In fact, there will always be an odd number of these small tricolored triangles.



A "Sapienti Sat" proof of Sperner's lemma

Brouwer's Fixed-Point Theorem (for $n = 2$). *Let $B_2 = \left\{x \in \mathbb{R}^2 : |x| \leq 1\right\}$. Any continuous mapping $f : B_2 \to B_2$ has a fixed point.*

Note: it suffices to prove that $f : \Delta \to \Delta$ has a fixed point, where Δ is anything which is homeomorphic to B_2 (meaning there is a continuous bijection from B_2 to Δ). In our case, we will use a triangle sitting in $\mathbb{R}^3$ with vertices $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$.

Exercises

1.1: Show that any sequence $(x_n)_{n=1}^{\infty}$ in $[0,1]$ has a subsequence which converges in $[0,1]$.

1.2: Show that for any irrational number α, the sequence $(n\alpha)_{n=1}^{\infty}$ gets arbitrarily close to the integers. More precisely, show that for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ for which the distance from $n\alpha$ to the nearest integer is less than ϵ. Can you use your proof to show that there are infinitely many rational numbers $\frac{p}{q}$ with $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$?

1.3: Let E be a set of positive real numbers, and define its sum as $\sum_{x\in E} x := \sup\{\sum_{x \in F} x : F \subset E,\ F \text{ is finite}\}$. Show that if E is uncountable then $\sum_{x\in E} x = \infty$.

2.1: Show that $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$.

2.2: 15 students join a summer course. Every day, 3 students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

3.1: Show that Theorem 2 is no longer true if we start with a sequence of only mn distinct real numbers (we still request an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$).

****3.2:** Formulate and prove a generalization of Sperner's lemma (to higher dimensions). Use your generalization to derive a proof for Brouwer's fixed point theorem in any dimension.

4.1: Let $\mathbb{S}^1$ be the unit circle and let $g : \mathbb{S}^1 \to \mathbb{R}$ be continuous. Show that $g(x) = g(-x)$ for some $x \in \mathbb{S}^1$. This is a special case of the Borsuk-Ulam theorem.

***4.2:** Let T be a triangulation of B_2 (the closed unit disk in $\mathbb{R}^2$) whose vertices which lie in $\mathbb{S}^1$ (the unit circle) are symmetric about the origin (i.e. if a vertex $v \in V(T) \cap \mathbb{S}^1$ then $-v \in V(T) \cap \mathbb{S}^1$). Note that the vertices in the interior of the unit disk need not be symmetric about the origin. Also note that an arc in the unit circle between two vertices counts as an edge in $E(T)$, and it must be the edge of some triangle in the triangulation. Let $L : V(T) \to \{-1, +1, -2, +2\}$ be a labeling of the vertices in T which is odd on $\mathbb{S}^1$ (i.e. $L(-v) = -L(v)$ for every $v \in V(T) \cap \mathbb{S}^1$). Show that T contains a complementary edge (an edge whose vertices are labeled with $-i$ and $+i$ for $i = 1$ or 2). *Hint:* Consider walking along some path in the dual graph of T. This result is called Tucker's lemma.

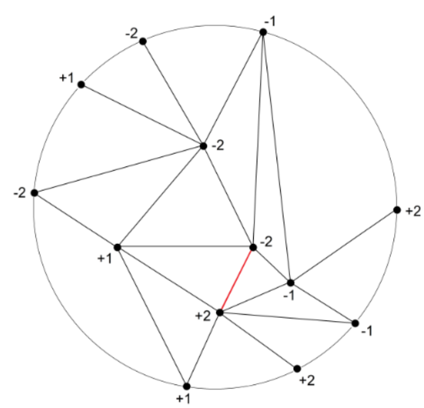


Image for Tucker's lemma

****4.3:** Can you generalize Tucker's lemma (to higher dimensions) and then use your generalization to prove a generalization of exercise 4.1 (the Borsuk-Ulam theorem)? *Hint:* If you're having trouble, try using the internet.

***5.1:** Prove any of the four theorems listed on page 1 which we didn't have time to prove in class.