Lec 9/26

Monday, September 26, 2016 8:07 AM

Moment generating function:

$$E(e^{tx}) = \sum_x e^{tx} P(X=x)$$

$$= \sum_x \left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right) P(X=x)$$

$$= \sum_x P(X=x) + \sum_x tx\, P(X=x) + \sum_x \frac{t^2x^2}{2!} P(X=x) + \cdots$$

$$= 1 + t\sum_x x P(X=x) + \frac{t^2}{2!}\sum_x x^2 P(X=x) + \cdots + \frac{t^i}{i!}\sum_x x^i P(X=x) + \cdots$$

$$= 1 + t E(X) + \frac{t^2}{2!} E(X^2) + \cdots + \frac{t^i}{i!} E(t^i) + \cdots$$

coefficient of $E(X^r) = \mu_r'$ is $\frac{t^r}{r!}$

$$\frac{\partial}{\partial t} M_X(t) = E(X) + t E(X^2) + \frac{t^2}{2!} E(X^3) + \cdots$$

so $\left.\frac{\partial}{\partial t} M_X(t)\right|_0 = E(X)$

$$\left.\frac{\partial^2}{\partial t^2} M_X(t)\right|_0 = E(X^2)$$

$$\left.\frac{\partial^r}{\partial t^r} M_X(t)\right|_0 = E(X^r) = \mu_r'$$

but this might not exist

Discrete example:

let X be a RV w pmf $p(x) = P(X=x) = \frac{e^{-x}\lambda^x}{x!}$

(poisson RV) for $x = 0, \dots$

Find the MGF

$$M_X(t) = e^{\lambda(e^t - 1)}$$

so $E(X) = M_X'(0) = e^{\lambda e^t - \lambda} \cdot \lambda e^t \Big|_0 = 1 \cdot \lambda = \lambda$

[Calculating $E(X)$ w/o mgf:

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \dots = \lambda$$

so $E(X^2) = M_X''(0) = e^{\lambda e^t - \lambda} \lambda e^t \lambda e^t + e^{\lambda e^t - \lambda} \lambda e^t \Big|_0$

$= \lambda^2 + \lambda$

so $Var(X) = \lambda^2 + \lambda - (\lambda)^2 = \lambda$

So $E(X) = Var(X)$ (special about poisson)

Theorem: if a, b constants, $b \neq 0$:

$$M_{X+a}(t) = E(e^{t(X+a)}) = e^{at} M_X(t)$$

$$M_{bX}(t) = E(e^{t(bX)}) = M_X(bt)$$

$$M_{\frac{X+a}{b}}(t) = E\left(e^{t\left(\frac{X+a}{b}\right)}\right) = e^{\frac{a}{b}t} M_X\left(\frac{t}{b}\right)$$

$\frac{X-\mu}{\sigma}$: z-score distribution