Lec 11/9

Wednesday, November 9, 2016 8:07 AM

$Y = u(X)$, $u(X)$ diff & either inc. or dec. $\Rightarrow$ u^{-1} exists.

Then $f_Y(y) = f_X(u^{-1}(y)) \left| \frac{d}{dy}(u^{-1}(y)) \right|$

Ex: let $Z \sim N(0,1) \Rightarrow f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

Define $Y = Z^2 \Rightarrow Z = \sqrt{Y}$ if $z > 0$.
$= -\sqrt{y}$ if $z < 0$.

$\frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$, $\frac{d}{dy}(-\sqrt{y}) = \frac{-1}{2\sqrt{y}}$

$$f_Y(y) = f_Z(\sqrt{y}) \left| \frac{1}{2\sqrt{y}} \right| + f(-\sqrt{y}) \left| \frac{-1}{2\sqrt{y}} \right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}} \cdot 2$$

$$= \frac{e^{-y/2}}{\sqrt{2\pi y}} \quad \text{for } y > 0$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}} y^{-1/2} e^{-y/2}$$

$$= \frac{1}{2^{1/2}\sqrt{\pi}} y^{\frac{1}{2}-1} e^{-\frac{y}{2}}$$

$$= \frac{1}{\Gamma(\frac{1}{2}) 2^{1/2}} y^{\frac{1}{2}-1} e^{-\frac{y}{2}}$$

so $Y \sim \text{Gamma}(\frac{1}{2}, 2)$

$Y \sim \chi^2_1$

7.4 Transformation Tech. multiple vars

Let $X \sim Po(\theta)$, $Y \sim Po(\lambda)$, X, Y independent.

$$f_{XY}(x,y) = \left(\frac{e^{-\theta}\theta^{x}}{x!}\right)\left(\frac{e^{-\lambda}\lambda^{y}}{y!}\right) \quad \text{for } x = 0, 1, 2, \dots \quad y = 0, 1, 2, \dots$$

let $U = X + Y$, $V = Y$ $\rightarrow$ if $Y = 12$ and want $P(U = 25)$, this is $P(X = 13)$

so $Y = V$, $X = U - Y = U - V$.

$V = 0, 1, 2, \dots$

$U = V, V+1, V+2, \dots$

$$f_{UV}(u,v) = f_{XY}(u-v, v) = \frac{e^{-\theta}\theta^{u-v}}{(u-v)!}\frac{e^{-\lambda}\lambda^{v}}{v!}$$

$$f_U(u) = \sum_{v=0}^{u} f_{XY}(u-v, v) = \sum_{v=0}^{u} \frac{e^{-\theta-\lambda}\theta^{u-v}\lambda^{v}}{(u-v)!\,v!}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!}\sum_{v=0}^{u}\frac{u!}{(u-v)!\,v!}\lambda^{v}\theta^{\theta+\lambda}$$

$$= \frac{e^{-(\theta+\lambda)}(\theta+\lambda)^{u}}{u!} \quad \text{binom. thm.}$$

So $U = Po(\theta + \lambda)$