Lec 11/4

Friday, November 4, 2016 8:13 AM

Functions of RVs

Ex:

1) $\bar{X} = \sum_{i=1}^{n} \frac{1}{n} X_i$

2) $S^2 = \frac{\Sigma (X_i - \bar{X})^2}{n-1}$

$= \frac{(\Sigma X_i^2) - n\bar{X}^2}{n-1}$

3) $X_{(n)} = \max\{X_1, \ldots, X_n\}$

$X_{(1)} = \min\{X_1, \ldots, X_n\}$

$R = X_{(n)} - X_{(1)}$

4) $Y = 2X + 5$

5) $Y = X^2$

3 methods:

1) Dist function technique

✓ 2) Transformation tech

3) MGF tech.

Ex: $X \sim f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$ Find PDF for $Y = X^2$.

Note: $0 < Y < 1$

Let $y \in (0, 1)$.

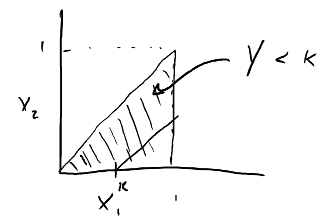
$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = \int_0^{\sqrt{y}} 2x\,dx = x^2\Big|_0^{\sqrt{y}} = y$

So $F_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 < y < 1 \\ 1 & y > 1 \end{cases}$

$f_Y(y) = F_Y'(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$ $\longrightarrow$ This should give a valid pdf.

$Y \sim Unif(0,1)$

Ex: let $f(x_1, x_2) = \begin{cases} 3x_1 & \text{if } 0 < x_2 < x_1 < 1 \\ 0 & \text{o.w} \end{cases}$



let $Y = X_1 - X_2$ Note: $0 < Y < 1$

Fix Y at k

$$F_Y(y) = P(Y \le y) = 1 - P(Y > y) = 1 - P(X_1 - X_2 > y) = 1 - P(X_2 < X_1 - y)$$

$$= 1 - \int_y^1 \int_0^{x_1 - y} 3x_1 \, dx_2 \, dx_1$$

$$= 1 - \int_y^1 (3x_1^2 - 3x_1 y) \, dx_1$$

$$= 1 - \left(x_1^3 - \frac{3}{2} x_1^2 y\right)\Big|_y^1 = 1 - \left(1 - \frac{3}{2}y - y^3 + \frac{3}{2}y^3\right)$$

$$= 1 - \left(1 - \frac{3}{2}y + \frac{1}{2}y^3\right)$$

$$= \frac{1}{2}(3y - y^3)$$

$f_Y(y) = F_Y'(y) = \begin{cases} \frac{3}{2}(1 - y^2) & 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$

Ex: Suppose X is a CRV w/ pdf $f(x)$

and $Y = X^2$

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(|X| \leq \sqrt{y}) = P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,dx = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

so $f_Y(y) = F_Y'(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\frac{-1}{2\sqrt{y}}$

$$= \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

Ex:

$X \sim \text{Unif}(0,1)$ $\quad Y = -\lambda \log(X)$ for $\lambda > 0$

$Y > 0$

for $y > 0$

$$F_Y(y) = P(Y \leq y) = P(-\lambda \log(X) \leq y)$$

$$= P(X \geq e^{-\frac{y}{\lambda}})$$

$$= 1 - P(X \leq e^{-\lambda/y})\ ?$$

$$= 1 - e^{-\lambda/y}$$

$$\Rightarrow Y \sim \text{Exp}\left(\tfrac{1}{\lambda}\right)$$