Lec 10/3

Monday, October 3, 2016 7:58 AM

4.6 Product moments

$Var(X-Y)$

$\vdots$

$E(XY)$ (product moment)

The r^{th} and s^{th} product moment of RVs X and Y

is $\mu'_{r,s} = E(X^r Y^s) = \sum_x \sum_y x^r y^s p(x,y)$

or $= \iint_{\mathbb{R}^2} x^r y^s f(x,y)\, dA$

The r^{th} and s^{th} product moment about the means of X and Y

is $\mu_{r,s} = E\left[(X-\mu_X)^r (Y-\mu_Y)^s\right]$ where $\mu_X = E(X)$, $\mu_Y = E(Y)$

$$= \sum_x \sum_y (x-\mu_x)^r (y-\mu_y)^s p(x,y)$$

or $$= \iint_{\mathbb{R}^2} (x-\mu_x)^r (y-\mu_y)^s f(x,y)\, dA$$

If $r=s=1$, then $\mu_{1,1} = E\left[(X-\mu_X)(Y-\mu_Y)\right]$

$$= E(XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y)$$

$$= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X \mu_Y$$

$$= E(XY) - \mu_X \mu_X$$

$$= E(XY) - E(X)E(Y)$$ ← covariance of X and Y

Covariance of X and Y is

$$\sigma_{X,Y} = Cov(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right] = E(XY) - E(X)E(Y)$$

$$Cov(X,X) = Var(X)$$

$$Cov(X,Y) = Cov(Y,X)$$

Ex: $f(x,y) = \begin{cases} 3x & 0 \le y \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$

$$g(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$h(y) = \begin{cases} \frac{3}{2}(1-y^2) & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(X) = \int_0^1 x\, 3x^2 \, dx = \frac{3}{4}$$

$$E(Y) = \int_0^1 y \, \frac{3}{2}(1-y^2)\, dy = \frac{3}{2}\int_0^1 y - y^3 \, dy = \frac{3}{8}$$

$$E(XY) = \int_0^1 \int_0^x xy \cdot 3x \, dy \, dx = \int_0^1 \left(\frac{3x^2 y^2}{2}\right)\Big|_0^x dx = \frac{3}{2}\int_0^1 x^4 \, dx = \frac{3}{10}$$

$$Cov(X,Y) = \frac{3}{10} - \frac{9}{32} = \frac{96}{320} - \frac{90}{320} = \frac{3}{160}$$

$Cov(X,Y) \in \mathbb{R}$

If $Cov(X,Y) < 0$, then X and Y tend to vary from their means in opposite directions.

If $Cov(X,Y) > 0$, then X and Y tend to vary from their means in the same direction.

Note: $Corr(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$\searrow \in [-1, 1]$

If $Cov(X,Y) = 0$, Then X and Y may be independent but $\mathrm{Cov}(X,Y) = 0 \not\Rightarrow$ Ind

(see example 4.17, pg 134)

Does Ind $\Rightarrow \mathrm{Cov}(X,Y) = 0$? Yes.

Let's suppose X, Y ind. and joint pdf is $f(x,y)$.

by defn., $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$$E(XY) = \iint_{\mathbb{R}^2} xy f(x,y)\, \partial A = \iint_{\mathbb{R}^2} xy\, g(x) h(x)\, \partial x\, \partial y = \int_{\mathbb{R}} x g(x)\, \partial x \int_{\mathbb{R}} y h(y)\, \partial y$$

$$= E(X)E(Y)$$

so $\mathrm{Cov}(X,Y) = 0$.

$E(XY) = E(X)E(Y)$ if X and Y are independent.