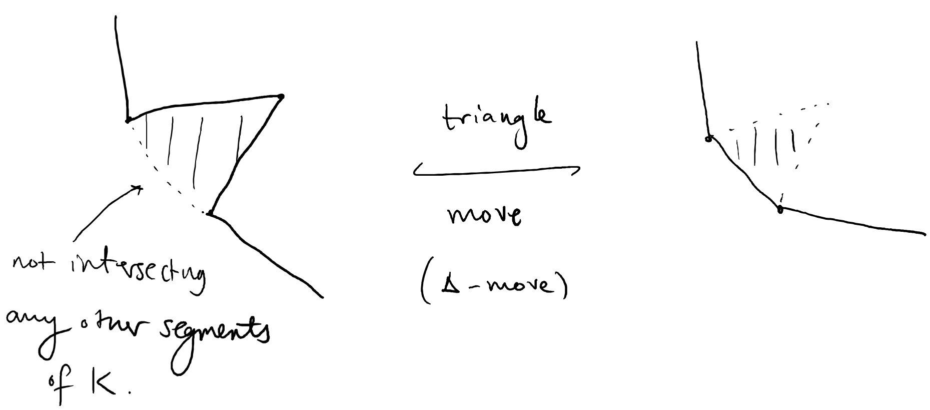
Thm Two knots $K_0$ & $K_1$ are equivalent

$\Leftrightarrow$ there is an orient. preserving homeom.

$f: S^3 \circlearrowleft$ with $f(K_0) = K_1$ preserving knot orientation.

$K$ is PL-knot




We say two PL-knots $K_{in}$ & $K_{fin}$ in $S^3$

are combinatorially equivalent iff they

are related by a sequence of Δ-moves:

$$K_{in} = K_0 \leftrightarrow K_1 \leftrightarrow \cdots \leftrightarrow K_N = K_{fin}$$

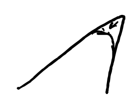
$$K_{in} \approx_{comb} K_{fin} \Longrightarrow K_{in} \approx K_{fin}$$

We say two smooth knots $K_0$ & $K_1$ are <u>smoothly equivalent</u> if there is a smooth isotopy between them.

$$F: S^1 \times [0,1] \longrightarrow S^3$$

$$K_0 \approx_\infty K_1 \Rightarrow K_0 \approx K_1$$

Suppose $K_0$ = $\varepsilon$-smoothing of $K_{in}$

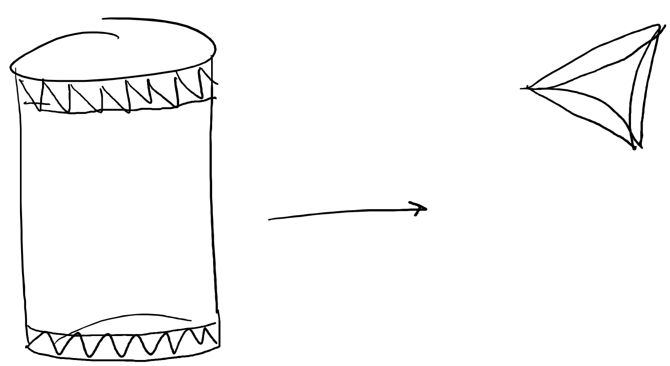


$K_1$ = " " $K_{fin}$

Then $K_{in} \approx_{comb} K_{fin} \Rightarrow K_0 \approx_\infty K_1$.

—— " —— $\Leftarrow$ —— " ——

also true



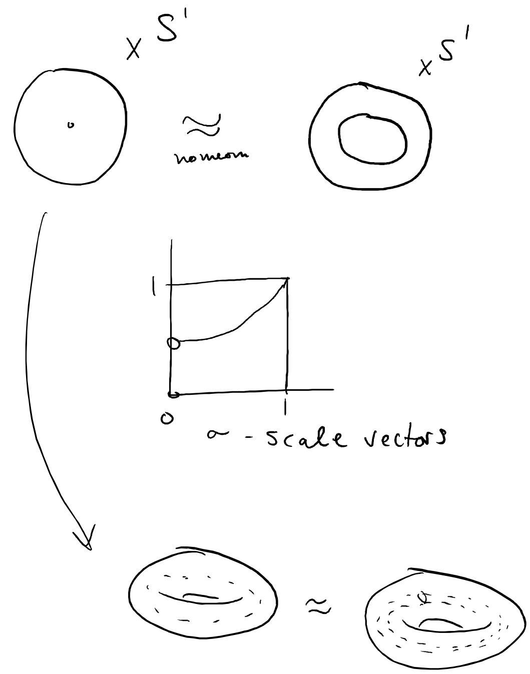
**Thm:** $\approx_{comb} = \approx_{\infty} = \approx$

Suppose $K_0$ & $K_1$ are PL-Knots.

Then the following are equivalent:

* $K_0 \approx K_1$
* $K_0 \approx_{comb} K_1$
* $K_0^{\varepsilon} \approx_{\infty} K_1^{\varepsilon}$ where $K_i^{\varepsilon}$ is some $\varepsilon$-smoothing of $K_i$.

## Basics of Knot/Link complements:




$\hat{\sigma} : (\vec{v}, \theta) = \left(\sigma(\|\vec{v}\|)\, \vec{v},\ \theta\right)$

Suppose $\tau : \mathring{D}^2_1 \times S^1 \hookrightarrow S^3$

is a tubular n.h. of $K$ (depending on framing).

$$(0 \times S^1 \longmapsto K)$$

$$\begin{tikzcd}
(\mathring{D}^2_1 \setminus 0) \times S^1 \arrow[r, "\tau"] \arrow[d, "\hat{\sigma}", "\wr"'] & S^3 \setminus K \arrow[d, "\wr"] \\
(D^2_1 - D^2_r) \times S^1 \arrow[r, "\tau"] & S^3 - \overline{N_r(K)}
\end{tikzcd}$$

— extend by id on $S^3 - N_1(K)$.

<u>Thm</u>: $S^3 \setminus K \approx S^3 \setminus \overline{N}_r(K)$

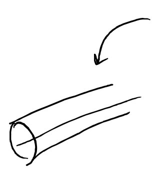
<u>Homology - considerations</u>

$S^3 - K$ $\qquad\qquad$ $M - K$

$M$ = homology 3-sphere

$H_0(M) = \mathbb{Z}, \quad H_1(M) = 0, \quad H_2(M) = 0, \quad H_3(M) = \mathbb{Z}.$

$$\mathring{N}(K) \quad \cup \quad (M - K) \quad = \quad M$$



for tub n.h. / frame $\tau$

$$\mathring{N}(K) \quad \cap \quad (M - K) \;\overset{\tau}{\text{———}}\; (\mathring{D}^2_1 - 0) \times S^1$$

$\simeq$ h.e.

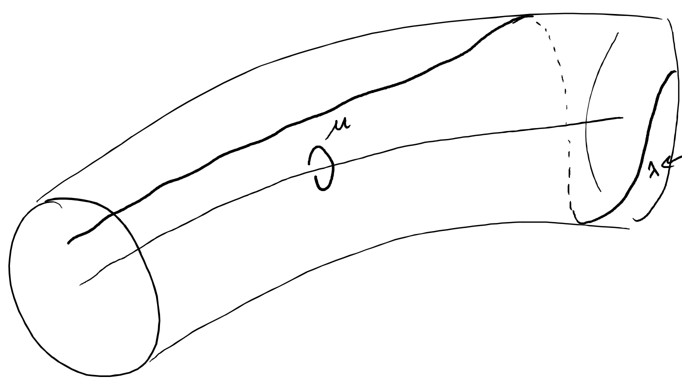
$S^1_m \times S^1_\ell$

In homology,

$$H_1(\dot{N}(K) \cap (M-K)) \xleftarrow[\cong]{\tau'_*} H_1(S^1_m \times S^1_\ell)$$

$$= \mathbb{Z}_{(\langle m \rangle)} \oplus \mathbb{Z}_{(\langle \ell \rangle)}$$

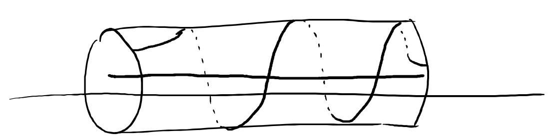
$$[\mu] = \tau'_*(\langle m \rangle) \qquad [\lambda] = \tau'_*(\langle \ell \rangle)$$



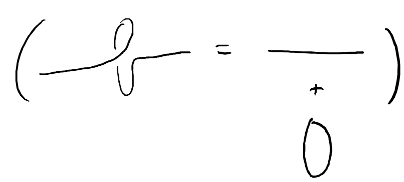

depends on framing implied by $\tau$.

If we change framing by $K \in \mathbb{Z}$,

we obtain $\lambda'$ from $\lambda$ (by adding $K$-twists).



$$[\lambda'] = [\lambda] + K[\mu].$$



$$H_1(\underset{\mathring{D}^2_1 \times S^1}{\mathring{N}(K)}) = \mathbb{Z}$$

$$\begin{tikzcd}
\mathring{N}(K) \cap (M \setminus K) \arrow[r, hook, "j"] \arrow[d, "H_1"'] & \mathring{N}(K) \arrow[d, "H_1"] \\
\mathbb{Z}[\mu] + \mathbb{Z}[\lambda] \arrow[r, "j_*"'] & \mathbb{Z}[\bar{\lambda}]
\end{tikzcd}$$

$$[\mu] \longmapsto 0$$

$$[\lambda] \longmapsto [\bar{\lambda}]$$

## Mayer-Vietoris

$$H_2(M) = 0 \longrightarrow H_1(\mathring{N}(K) \cap (M - K)) \xrightarrow{\text{isomorphism}} H_1(\mathring{N}(K)) \oplus H_1(M-K) \longrightarrow 0 = H_1(M)$$

$$\mathbb{Z}_{([\mu])} \oplus \mathbb{Z}_{([\lambda])} \underset{L}{\xrightarrow{\cong}} \mathbb{Z}_{([\bar{\lambda}])} \oplus H_1(M-K)$$

$$H_1(M-K) \cong \mathbb{Z}_{([q])}$$

$$([\mu], [\lambda]) \qquad\qquad ([\bar{\lambda}], [q])$$

$$L = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix}, \qquad L : \mathbb{Z}^2 \xrightarrow{\cong} \mathbb{Z}^2$$

$$\alpha = \pm 1$$

$\alpha = 1$ for one of gen

$$L = \begin{bmatrix} 0 & 1 \\ 1 & \beta \end{bmatrix} \qquad \begin{array}{l} [\lambda] \longmapsto [\bar{\lambda}] + \beta [\eta] \\ [\mu] \longmapsto [\eta] \end{array}$$

depends on framing

$$[\lambda'] = [\lambda] - \beta [\mu] \longrightarrow [\bar{\lambda}] + \beta [\eta] - \beta [\eta] = [\bar{\lambda}].$$

<u>Lemma</u>: There is exactly one framing s.t.

for $j : (\mathring{N}(K) - K) \hookrightarrow (M - K)$,

$j_* ([\lambda]) = 0$ .

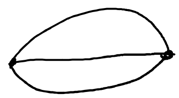
$\Rightarrow$ $M$ = homology-sphere, 0-framing well-defined.

$*$ this is, for example, the case if respective longitude bounds a 2d oriented surface $\Sigma \subset M - K$

<u>Exercise</u>: find a longitude for trefoil that represents 0-framing.

$\Sigma_{g,n}$

$\Sigma_{1,1}$



"push the knot in a little bit to get the longitudinal curve"

link

$$H_1(M - \mathcal{L}) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k \,=\, \#\text{ components of } \mathcal{L}}$$ (Exercise: prove this).