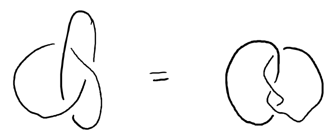
8/23

Friday, August 23, 2019 15:00

<u>Regularity</u>

$i : S^1 \hookrightarrow S^3$

$i_0 , i_1 : S^1 \hookrightarrow S^3$



$i_t : S^1 \hookrightarrow S^3$

<u>Isotopy</u>: $I = [0,1]$

* level-preserving embedding
$$\begin{tikzcd} X \times I \arrow[rr, "\hat{f}"] \arrow[dr] & & Y \times I \arrow[dl] \\ & I & \end{tikzcd}$$

* $\hat{f}(x,t) = (f(x,t), t)$ , $f : X \times I \to Y$

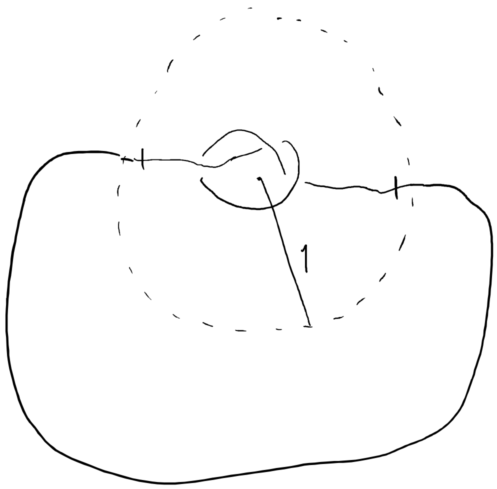
$(f \to \hat{f}$ trace$)$

* $f_t(x) = f(x,t)$

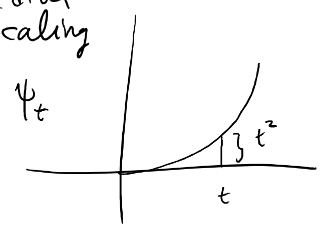
<u>Def</u> $g, h : X \to Y$ are isotopic if $\exists$ isotopy

$$f : X \times I \to Y \quad \text{s.t.} \quad f_0 = g, \; f_1 = h.$$

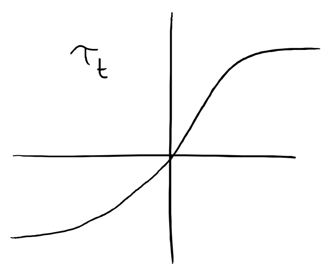
<u>Lemma</u>: many knots are isotopic.



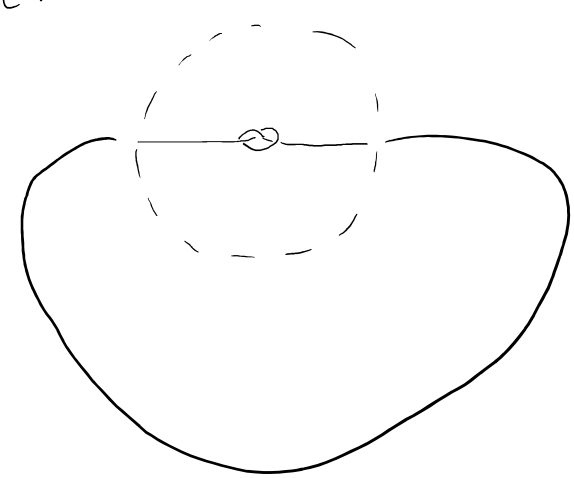

radial
scaling




timescaling

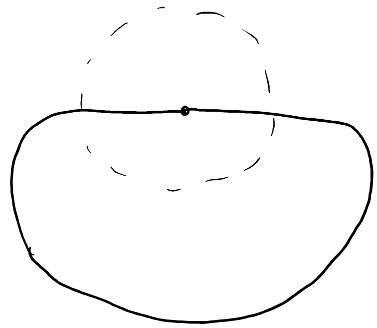



$t > 0$



$$i_t(\theta) = \psi_t\left(i(\tau_t(\theta))\right)$$

$t = 0$:



$i_t \; S^1 \hookrightarrow \mathbb{R}^3$ smooth map $\forall t$

$i_0$ represents unknot,

$i_t$ = trefoil $\forall t > 0$.

SO: regular isotopy is not good enough.

<u>Def</u> two embeddings $f_0, f_1 : X \hookrightarrow Y$

are <u>ambient isotopic</u> if there is an isotopy

$$\hat{\Phi} : Y \times I \longrightarrow Y \times I$$

s.t. $\Phi_0 = id_Y$, $\Phi_t$ = homeomorphism $\forall t$,

and

$$\begin{tikzcd} X \arrow[r, hook, "f_0"] \arrow[rr, bend right, "f_1"'] & Y \arrow[r, "\Phi_1"] & Y \end{tikzcd} \qquad \left( f_1 = \Phi_1 \circ f_0 \right)$$

$\Longrightarrow \quad f_t = \Phi_t \circ f_0$ gives a regular isotopy.

<u>Def</u> two knots are equivalent if they are ambient isotopic.

$i_0, i_1 : S^1 \hookrightarrow S^3, \quad K_j = i_j(S^1) \subset S^3$

This implies $\Phi_1 : S^3 \to S^3$

s.t. $\Phi_1(K_0) = K_1$

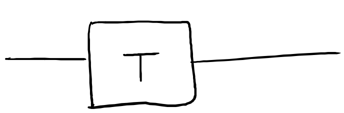
$\Phi_1 : S^3 \setminus K_0 \longrightarrow S^3 \setminus K_1$

also a homeomorphism.

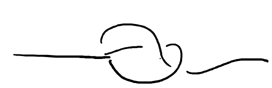
$\pi_1(S^3 \setminus K)$ is the knot group of $K$.

$\Phi_1^* : \pi_1(S^3 \setminus K_0) \longrightarrow \pi_1(S^3 \setminus K_1)$ is an isomorphism.

<u>Eg</u> $\pi_1(S^3 \setminus O) = \mathbb{Z}$.

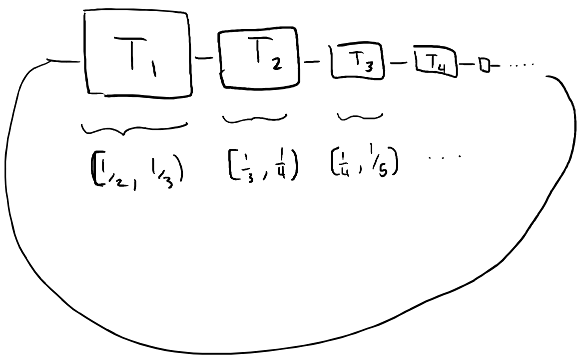



= e.g.



$\subset \mathbb{R}^3$

$T_1, T_2, \ldots$



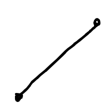

is a legit knot thus far.

<u>Def</u> piecewise-linear knot (PL) is PL embedding of $S^1 = [0,1]/_{0\sim 1}$ into $\mathbb{R}^3$

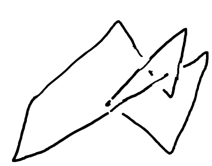
$$0 = s_0 < s_1 < \cdots < s_n = 1$$

$$i|_{[s_k, s_{k+1}]} \longrightarrow \mathbb{R}^3$$

is affine linear:



$$\vec{A}_k s + \vec{B}_k$$



<u>Def</u> call a knot "tame" if it is equivalent to a PL-knot.

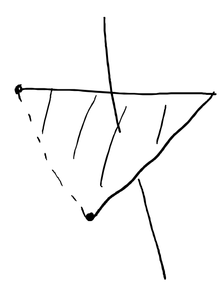
otherwise it's "wild".

$K$ : knot

$\text{stick}(K)$ = minimal # of sticks that can be used to make a PL-knot eq. to $K$.

$\text{stick}(K) \leq 5 \implies K \approx$ unknot

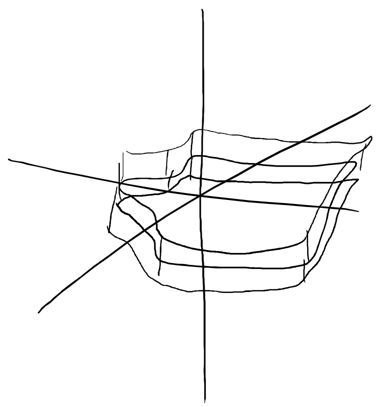
$\text{stick}(K) = 6 \implies K \approx$ trefoil



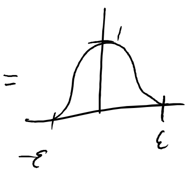
only two more sticks to put in, has to be unknot.

<u>Lemma</u> Suppose $\hat{h} : \mathbb{R}^2 \times I \longrightarrow \mathbb{R}^2 \times I$ is an ambient isotopy <u>supported</u> in an open n.h. $V \subset \mathbb{R}^2$.
Then $\hat{h}$ extends to ambient isotopy supported in $V \times (-\varepsilon, \varepsilon)$ given any $\varepsilon > 0$.

$\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$



$\rho(z) =$




$$H : \mathbb{R}^3 \times J \longrightarrow \mathbb{R}^3$$

$$H((\underset{\substack{\uparrow \\ \mathbb{R}^2}}{x}, z), s) = (h(\vec{x}, \rho(z)s), z)$$

## Regularity

Smooth embeddings $f : N \xrightarrow{C^\infty} M$

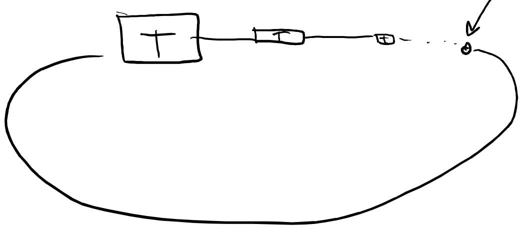
* top. embedding
* immersion. (That is, $Df(x)$ = injective)

Def smooth knot is smooth embedding

$$\gamma : S^1 \hookrightarrow S^3$$

$$\left( \frac{d\gamma}{dt} = \text{nowhere zero} \right)$$

$\frac{d\gamma}{dt}$ must be 0 here



cannot be constructed as smooth knot.

All PL-knots are ambient isotopic to smooth knots: