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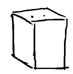
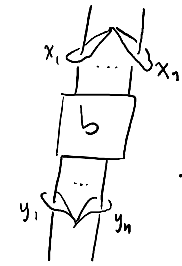
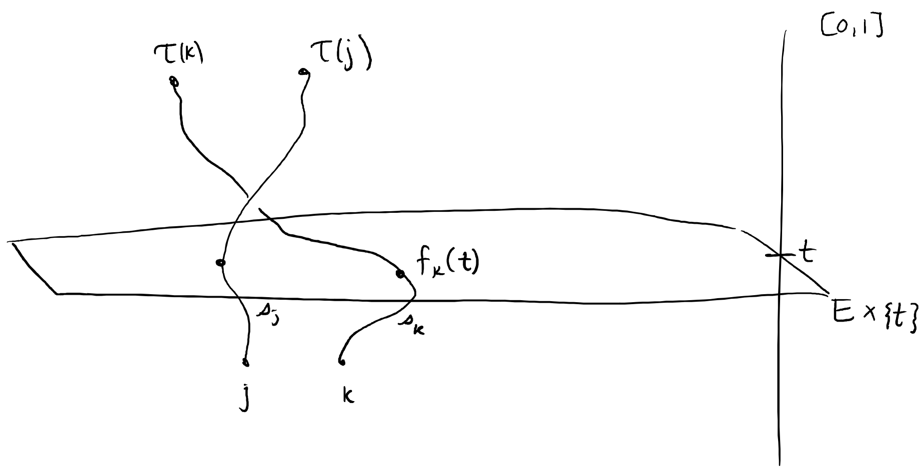
Monday, October 14, 2019 14:58

Recall: $A =$  $\setminus b$, $\quad L = cl(b)$

$\Rightarrow \quad \pi_L = \pi_1(A) / \langle x_i y_i^{-1} \rangle$ where x_i, y_i are given as in 

Now what is $\pi_1(A)$?



$(s_j)_3$ = strictly increasing

So assume $s_j(t) = (f_j(t), t)$,

$$f_j : [0,1] \longrightarrow E.$$

$$\varphi^b : P_n \times [0,1] \longrightarrow E \qquad \text{where } P_n = \{r_1, \dots, r_n\} \subset E.$$

$$(r_j, t) \longmapsto f_j(t)$$

$\varphi_t^b : P_n \longrightarrow E$ is an embedding.

By Isotopy Extension,

$$\phi^b : E \times [0,1] \longrightarrow E$$

$$\phi_0^b = \mathrm{Id}, \quad \phi_t^b = \text{diffeom } \forall t,$$

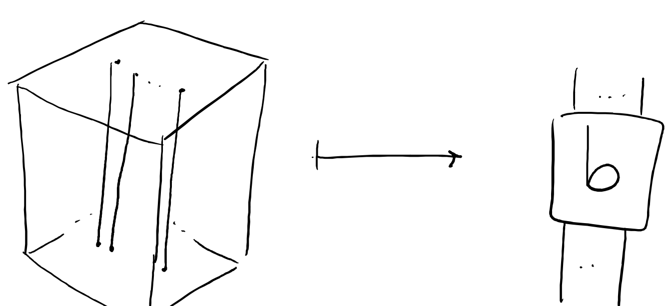
$$\phi_t^b(r_j) = \varphi_t^b(r_j) = f_j(t)$$

$$\hat{\phi}^b : E \times [0,1] \longrightarrow E \times [0,1]$$

$$(z, t) \longmapsto (\phi_t^b(z), t)$$

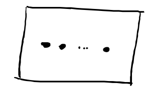
$$\hat{\phi}^b : P_n \times [0,1] \longrightarrow E \times [0,1]$$

$$(r_j, t) \longmapsto (f_j(t), t)$$



$\hat{\phi}^b : E \times [0,1] \to$

$$\hat{\phi}^b : E \times [0,1] \setminus P_n \times [0,1] \longrightarrow E \times [0,1] \setminus b = A$$

Important map: $\phi_1^b : E^* \longrightarrow E^*$ $\qquad E^* =$ 

$$\begin{tikzcd}
z_i \arrow[d, mapsto] \arrow[rrr, bend left, mapsto] & \pi_1(E^*) \arrow[d, "\text{top}"] & \pi_1(E^*) \arrow[l, "\rho_b = (\hat{\phi}_1^b)_*"'] \arrow[d, "(-,1)_*"] & \rho_b^{-1}(z_i) \\
y_i & \pi_1(A) & \pi_1(E^* \times [0,1]) \cong \pi_1(E^*) = \langle z_1, \dots, z_n \rangle \arrow[l, "(\hat{\phi}^b)_*"'] & \\
x_i & & & \\
z_i \arrow[u, mapsto] \arrow[rrr, bend right, mapsto] & \pi_1(E^*) \arrow[uu, "\text{bottom}"'] & \pi_1(E^*) \arrow[l, "\text{id}"] \arrow[uu, "(-,0)_*"'] & z_i
\end{tikzcd}$$

So $\pi_L \cong \langle \{z_i\} \mid \{\rho_b^{-1}(z_i)\, z_i^{-1}\} \rangle$

$\cong \langle \{w_i\} \mid \{\rho_b(w_i)\, w_i^{-1}\} \rangle$

Exercises (see notes)

- $\hat{\phi}, \hat{\psi} : E^* \times [0,1] \longrightarrow A$ level pres.

id on $E^* \times 0$

then $(\phi_1)_* = (\psi_1)_*$

- if $b \sim b'$ then $\rho^b \approx \rho^{b'}$

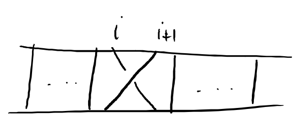
- if $b = b_2 \circ b_1$,

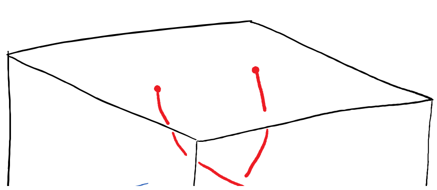
$$\rho^b = \rho^{b_2} \circ \rho^{b_1}$$

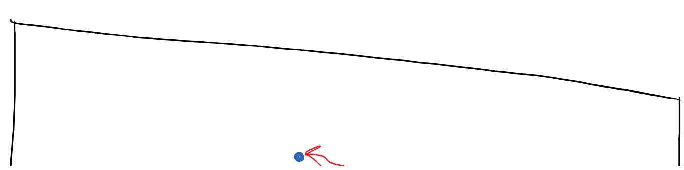
Thus $[b] \longmapsto \rho^b$ yields a well-defined homomorphism $B_n \longrightarrow \mathrm{Aut}(\pi_1(E_n^*))$.

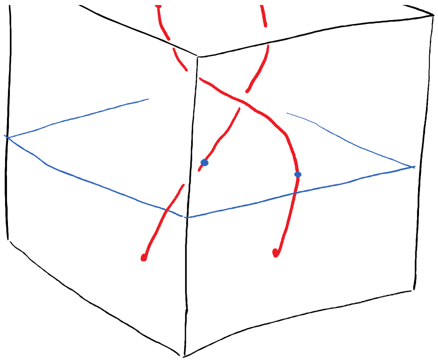
That is, $B_n \circlearrowright \pi_1(E_n^*) = \langle z_1, \dots, z_n \rangle$

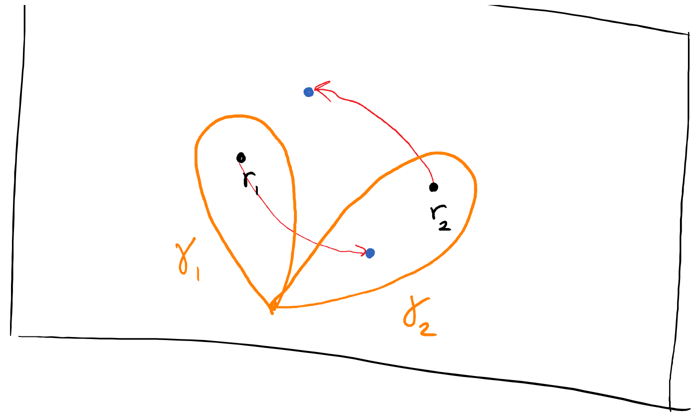
$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \ \text{ if } |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle$$

$\sigma_i =$  .

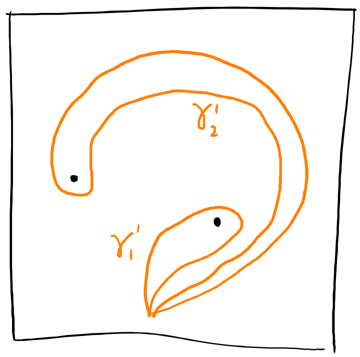




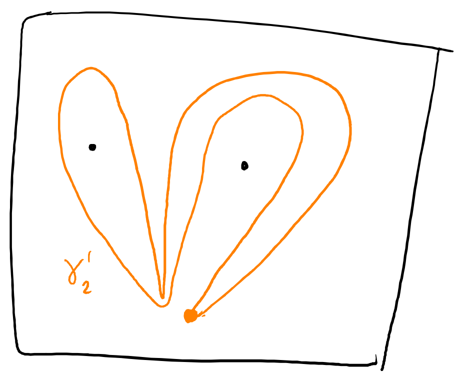




$[\gamma_i] = z_i$



$\gamma_i' = \phi_b' \circ \gamma_i$



$[\gamma_1'] = z_2,$

$[\gamma_2'] = [\gamma_2^{-1} * \gamma_1 * \gamma_2]$

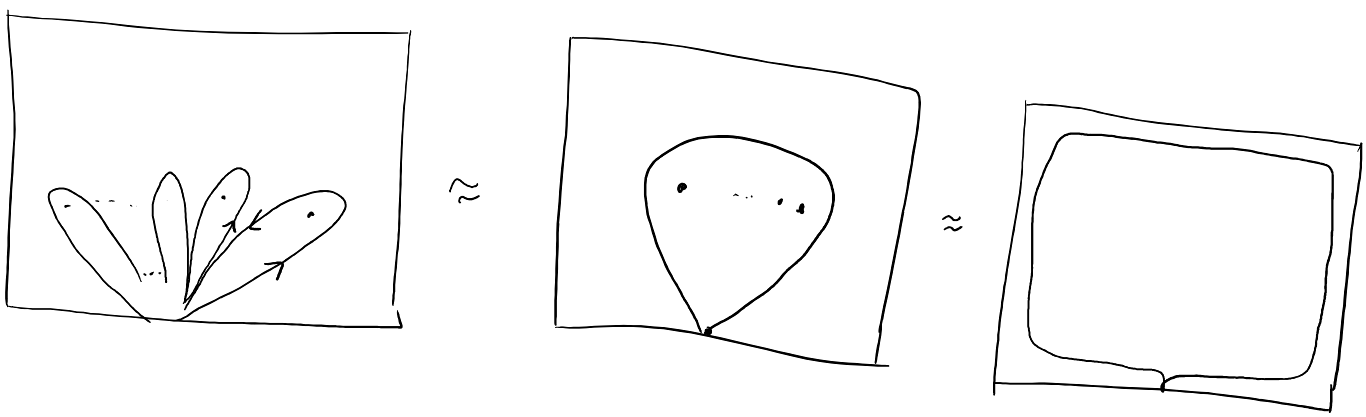
$= z_2^{-1} z_1 z_2$

Action of B_n on $F_n = \langle z_1, \ldots, z_n \rangle$ given by

$$\sigma_i \cdot z_j = \begin{cases} z_j & \text{if } j \notin \{i, i+1\} \\ z_{i+1} & \text{if } j = i \\ z_{i+1}^{-1} z_i z_{i+1} & \text{if } j = i+1 \end{cases}.$$

In Wirtinger repn, One relation is redundant:

Let $\quad z_\infty = z_1 \cdots z_n$



So $\quad b \cdot z_\infty = z_\infty \overset{\text{exercise}}{\Longrightarrow}$ one relation is redundant here too.

Trefoil $=$ cl().

$$\mathrm{Hom}(\pi_1(S_k), G)$$

$$\mathrm{Hom}(\underbrace{\pi_1(E \setminus \{a,b\})}_{F_2}, G)$$

$$= \mathrm{Hom}(F_2, G) = G \times G$$

$\sigma \in B_2$

acts on $G \times G$ and on $C(G \times G) = C(G) \otimes C(G) \xrightarrow{R} C(G) \otimes C(G)$.

$$M_n = \mathrm{Diffeom}(E_n^*, \overset{\text{rel}}{\downarrow}\partial E)$$

$$\Gamma(E_n^*, \partial E) = M_n / \pi_0(M_n)$$

$=$ Mapping Class group of E_n^*

<u>Prop</u> $B_n \longrightarrow \Gamma(E_n^*, \partial E)$ is an isomorphism.

$b \longmapsto [\hat{\phi}_1^b]$

Artin Representation Theorem

$$F_n = \langle z_1, \dots, z_n \rangle, \quad \underbrace{\overline{B}_n}_{\text{image of } B_n} < Aut(F_n)$$

Then $\overline{B}_n$ consists precisely of elements $\alpha \in Aut(F_n)$ for which $\exists\ \tau \in S_n$ and $A_j \in F_n$ $(j = 1, \dots, n)$ w/

$$\alpha(z_j) = A_j \, z_{\tau(j)} \, A_j^{-1} \quad \underline{\text{and}} \quad \alpha(z_\infty) = z_\infty .$$

Alexander Module

$$\underbrace{\pi_K \rhd \pi_K' \rhd \pi_K^{(2)} \rhd \pi_K^{(3)} \rhd \cdots}_{\text{Derived series}}$$

$\left(\pi_K^{(i)} = [\pi_K^{(i-1)}, \pi_K^{(i-1)}] \right)$ Look at $\pi_K^{(1)} / \pi_K^{(2)}$