O. Costin Stokes Phenomenon Sept 25^{th} W 4:15-5:15 pm MW154.

<u>several variables</u> $F(z_1, \dots, z_n)$

$$\frac{\partial F}{\partial z_j} = A_j(z_1, \dots, z_n)\, F(z) \qquad (\forall\, 1 \le j \le n) \qquad (*)$$

$\mathcal{U}$ = a unital associative algebra over $\mathbb{C}$ (eg $M_{N\times N}(\mathbb{C})$)

$A_1, \dots, A_n : D \longrightarrow \mathcal{U}$ are $\mathcal{U}$-valued holomorphic fns.

<u>Defn</u> $(*)$ is said to be consistent (or integrable, or flat) if $\forall j \ne k$, $\frac{\partial}{\partial z_k} A_j - \frac{\partial}{\partial z_j} A_k + [A_j, A_k] = 0$.

<u>Theorem</u> $(*)$ has an invertible solution $\Leftrightarrow$ $(*)$ is consistent.

<u>pf</u> (Necessity). Let $G(z)$ be a solution.

Then
$$\frac{\partial^2}{\partial z_j \partial z_k} G = \frac{\partial}{\partial z_j}(A_k \cdot G) = \left(\frac{\partial}{\partial z_j} A_k\right) \cdot G + A_k A_j G$$
$$= \left(\frac{\partial}{\partial z_k} A_j\right) \cdot G + A_j A_k G$$

(sufficiency). $(n=2)$

$$\frac{\partial F}{\partial z_1} = A_1(z_1, z_2)\, F \quad \Big| \quad \frac{\partial F}{\partial z_2} = A_2(z_1, z_2)\, F$$

fix z_2 and solve the one var. problem: $\Psi(z_1, z_2)$

- Find $C(z_2)$ s.t. $G = \Psi \cdot C$ solves 2^{nd} eq^n.

$$\frac{\partial}{\partial z_2}\left(\Psi(z_1, z_2)\, C(z_2)\right) = A_2(z_1, z_2)\, \Psi(z_1, z_2)\, C(z_2)$$

$$\Leftrightarrow \quad C'(z_2) = \underbrace{\Psi^{-1} \cdot (\partial_{z_2} \Psi - A_2 \Psi)}\cdot C(z_2)$$

again solvable $\Leftrightarrow$ $\uparrow$ is independent of z_1 $\Leftrightarrow$ $\partial_{z_1}\left(\Psi^{-1}(\partial_{z_2}\Psi - A_2\Psi)\right) = 0$

$$\Leftrightarrow \quad -\Psi^{-1}(\partial_{z_1}\Psi)\,\Psi^{-1}(\partial_{z_2}\Psi - A_2\Psi) + \Psi^{-1}\left(\partial_{z_1}\partial_{z_2}\Psi - \partial_{z_1}(A_2\Psi)\right) = 0$$

$$\Leftrightarrow \quad A_1(\partial_{z_2}\Psi - A_2\Psi) + \partial_{z_2}(A_1\Psi) - \partial_{z_1}(A_2\Psi) = 0$$

$$\Leftrightarrow \quad -A_1(\partial_{z_1}\Psi) + A_1 A_2 \Psi + (\partial_{z_2} A_1)\Psi + A_1(\partial_{z_2}\Psi) - (\partial_{z_1} A_2)\Psi - A_2 A_1 \Psi = 0$$

multiply on right by Ψ^{-1}.

$\blacksquare$

(induct using this strategy to prove sufficiency for general n).

Language of differential forms $\quad \nabla \cdot F = 0$

$\Gamma(D;\mathcal{U}) :=$ all holomorphic functions $D \to \mathcal{U}$.

$\Omega^1(D;\mathcal{U}) :=$ 1-forms ($\mathcal{U}$-valued) on D $\quad \left(\Omega^1 = \bigoplus_{j=1}^n \Gamma\right)$

$\ni \sum_{j=1}^n f_j(z)\, dz_j$

think gradient: $f(z) \xrightarrow{d} \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j$

De Rham differential

Given $\mathcal{A} \in \Omega^1(D;\mathcal{U})$,

$\nabla = d - \mathcal{A}$ defines the system of PDEs

$\nabla F = 0 \iff dF = \mathcal{A} F$.

$\Omega^k(D;\mathcal{U}) :=$ k-forms. $dz_i \wedge dz_j = -dz_j \wedge dz_i$

$\| $

$\left\{ \sum_{1 \le i_1 < \cdots < i_k \le n} f_{\underline{i}}(z)\, dz_{i_1} \wedge \cdots \wedge dz_{i_k} \;\middle|\; f_{\underline{i}} \in \Gamma \right\}$ - rank $\binom{n}{k}$ module over Γ.

$$\Omega^1 \xrightarrow{d} \Omega^2 : \sum_{i=1}^n f_i(z) dz_i \overset{d}{\longmapsto} \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \right) \wedge dz_i = \sum_{1 \le j_1 < j_2 \le n} \left(\frac{\partial f_{j_2}}{\partial z_{j_1}} - \frac{\partial f_{j_1}}{\partial z_{j_2}} \right) dz_{j_1} \wedge dz_{j_2}.$$

<u>Lemma</u>: $\underbrace{\nabla F = 0}_{\nabla = d - A}$ is consistent $\iff d\mathcal{A} - \mathcal{A} \wedge \mathcal{A} = 0$

<u>Pf</u> $\mathcal{A} = \sum_{j=1}^n \mathcal{A}_j dz_j$. $\quad d\mathcal{A} = \sum_{1 \le j_1 < j_2 \le n} \left(\frac{\partial A_{j_2}}{\partial z_{j_1}} - \frac{\partial \mathcal{A}_{j_1}}{\partial z_{j_2}} \right) dz_{j_1} \wedge dz_{j_2}$

$\left(\sum_{i=1}^n \mathcal{A}_i dz_i \right) \wedge \left(\sum_{i=1}^n A_j dz_j \right) \longrightarrow$ coeff of $dz_{j_1} \wedge dz_{j_2}$ gives consistency condition ∎

eg $n = 2$, $\quad x, y$

$$\nabla = d - \left(\frac{dx}{x} t_1 + \frac{dy}{y} t_2 + \frac{d(x+y)}{x+y} t_3 \right)$$

$$\frac{\partial F}{\partial x} = \left(\frac{t_1}{x} + \frac{t_3}{x+y} \right) F, \quad \frac{\partial F}{\partial y} = \left(\frac{t_2}{y} + \frac{t_3}{x+y} \right) F$$

$t_1, t_2, t_3 \in \mathcal{U}$ eg $N \times N$ matrices.

HW: consistent iff $t_1 + t_2 + t_3$ commutes w/ t_1, t_2, & t_3.

PDEs with regular singularities along hyperplanes.

V: n-dim $\mathbb{C}$-v.s., $V^* =$ linear dual $= \mathrm{Hom}(V, \mathbb{C})$

- Let $X \subset V^*$ finite set s.t.

  (i) $0 \notin X$,

  (ii) $x \neq y$; $x, y \in X \Rightarrow x$ & y are not proportional

- for $x \in X$, let $t_x \in \mathcal{U}$.

  PDE: $\nabla F = 0$ where $\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x$

- Pick a basis $\{x_1, \ldots, x_n\}$ of V^*, $x = \sum_{i=1}^n n_i(x) x_i$. $\quad \iff \frac{\partial F}{\partial x_i} = \sum_{x \in X} \frac{n_i(x)}{x} t_x \cdot F \quad$ for $i = 1, \ldots, n$.

<u>Kohno's Lemma</u>: $\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x$ is consistent

$\iff$ for every $Y \subset X$ maximal so that $\dim(\mathrm{Span}\, Y) = 2$,

we have $\sum_{y \in Y} t_y$ commutes w/ $t_{y'}$ $\forall y' \in Y$.