9/6

Friday, September 6, 2019 13:51

$$\frac{dF}{dz} = A(z) F(z) \qquad (*)$$

Irregular singularities

$$A(z) = \sum_{k=-r-1}^{\infty} A_k z^k$$

$r =$ Poincare rank of irregular singularity.

We will assume $r = 1$.

$$A(z) = z^{-2} \Lambda + z^{-1} \underset{\in M_{N\times N}(\mathbb{C})}{X} + \sum_{k=0}^{\infty} A_k x^k$$

assume $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $\lambda_i \neq \lambda_j$ for $i \neq j$.

set $X^0 =$ diagonal part of X

$$X^{o\text{-}d} = X - X^0$$

$$z^{-2}\Lambda = \frac{d}{dz}(-z^{-1}\Lambda)$$

Theorem $\exists!\ Y(z) = 1 + \sum_{k \geq 1} Y_k z^k \in GL_N(\mathbb{C}[[z]])$

s.t. $\underset{\text{formal}}{\Psi(z)}$ $Y(z) \cdot z^{X^0} \cdot e^{-\Lambda/z}$ solves $(*)$

Proof $Y'(z) \cdot z^{X^0} e^{-\Lambda/z} + Y(z) \cdot \frac{X^0}{z} z^{X^0} e^{-\Lambda/z}$

$+ Y(z) \cdot z^{X^0} \cdot \frac{\Lambda}{z^2} e^{-\Lambda/z}$

$$= \left(\frac{\Lambda}{z^2} + \frac{X}{z} + A_{reg}(z) \right) \cdot Y(z) \cdot z^{X^\circ} \cdot e^{-\Lambda/z}$$

So
$$\frac{d}{dz} Y(z) = z^{-2} [\Lambda, Y(z)] + z^{-1} [X^\circ, Y(z)] + z^{-1} X^{o\text{-}d} Y(z) + A_{reg}(z) Y(z)$$

Compare coeffs:

$$z^{-2}: \quad 0 = [\Lambda, \overset{1}{\overset{\shortparallel}{Y_0}}] = 0 \quad \checkmark$$

$$z^{-1}: \quad 0 = [\Lambda, Y_1] + X^{o\text{-}d}$$

$$\vdots$$

Can get Y_n inductively $\square$

Example $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

$Y(z)$ solves.

$$Y(z) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$Y'(z) = z^{-2} \begin{bmatrix} 0 & \beta(z) \\ -\gamma(z) & 0 \end{bmatrix} + z^{-1} \begin{bmatrix} -\gamma(z) & -\delta(z) \\ 0 & 0 \end{bmatrix}$$

$\gamma \equiv 0$

$\alpha \equiv 1$

$\delta \equiv 1$

$$\beta'(z) = \frac{\beta(z)}{z^2} - \frac{1}{z}$$

$$\beta(z) = \sum_{n \geq 1} b_n z^n$$

$b_1 = 1$

$b_2 = b_1 = 1$

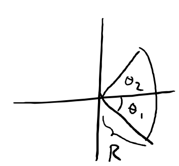
$2b_2 = b_3 \Rightarrow b_3 = 2$

$b_n = (n-1)!$

$$\underbrace{\beta(z) = \sum_{n \geq 1} (n-1)!\, z^n}_{\text{has } 0 \text{ radius of convergence.}}$$

$\sum_{n=1}^{\infty} z^n (n-1)!$ (Divergent Series by Hardy §2.4 - Euler &

Asymptotic expansions



Defn: for $R > 0$, $-\pi < \theta_1 \leq \theta_2 < \pi$,

$$S(R; \theta_1, \theta_2) = \{ z \in \mathbb{C} \mid |z| < R,\ \arg z \in (\theta_1, \theta_2) \}.$$

$\theta_1 < \theta_2$

Let $f(z)$ be a holomorphic fn on $S(R;\theta_1,\theta_2)$

Then $f(z) \sim \sum_{k=0}^{\infty} a_k z^k$ as $z \to 0$; $z \in S(R;\theta_1,\theta_2)$

<u>means</u>: $\forall M \geq 0$,

$$\lim_{\substack{z \to 0 \\ z \in S}} \left(f(z) - \sum_{k=0}^{M-1} a_k z^k \right) \cdot z^{-M} \text{ exists } (= a_M).$$

Some properties: (Wasow Asymptotic Expansion for ODE's Ch III)

(1) Term-wise addition, multiplication (even composition) is legitimate

(2) $f(x) \sim \sum_{k=0}^{\infty} a_k x^k \Rightarrow \int_0^x f(t)\,dt \sim \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{k+1}$

(as $x \to 0$, $x \in S$)

(same for differentiation $\theta_1 < \theta_2$)

(3) There are infinitely many functions asymptotic to the same series.

$$e^{-\frac{1}{x}} \sim 0 \qquad \left(x \to 0,\ x \in S(1; \tfrac{\pi}{4}, \tfrac{\pi}{4})\right).$$

(§9)

(4) J.F. Ritt (1914). Given a formal series $\sum_{}^{\infty} a_k z^k$

$k=0$

and $S(R;\theta_1,\theta_2)$, there exists $f(z)$ holomorphic on S s.t. $f(z) \sim \sum_{k=0}^{\infty} a_k z^k$ as $z \to 0$, $z \in S$.

Back to our ODE:

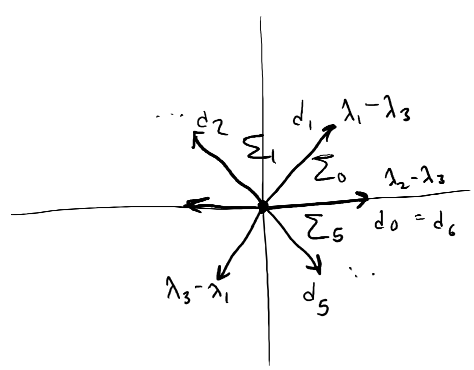
(Boalch. Stokes phenomenon, Poisson-Lie groups, Frobenius mfds, (Invent. Math. 2001))

$$A(z) = \left(\frac{\Lambda}{z^2} + \frac{X}{z}\right).$$

We know $\exists!$ formal solution $\Psi_{formal}(z) = Y(z) \cdot z^{X^0} e^{-\Lambda/z}$

Def An anti-stokes ray is a ray in $\mathbb{C}$ of the form $(\lambda_i - \lambda_j)\mathbb{R}_{>0}$ (for some $i \neq j$)

eg $\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} = \Lambda$



Let $2l$ = # of Stokes rays

Let $d_0, d_1, \dots, d_{2\ell-1}, d_{2\ell} = d_0$ be

an ordering of anti-Stokes rays (counter-clockwise)

Σ_i = sector bounded by d_i & d_{i+1}.

Thm (Sibuya). For each $i \in \{0, \dots, 2\ell-1\}$,

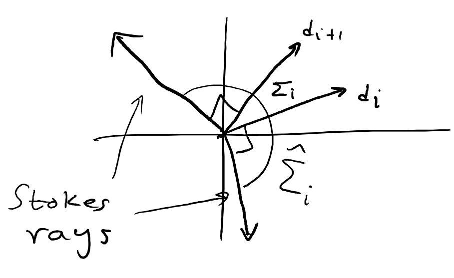
there is a unique holomorphic fn $Y_i(z)$ on Σ_i s.t.

(i) $Y_i \sim Y(z)$ as $z \to 0$ in Σ_i

(ii) $Y_i(z) z^{X^\circ} e^{-\frac{\Lambda}{z}}$ solves $(*)$

Extended sector

$\hat{\Sigma}_i$ bounded b/w $d_i - \pi/2$ and $d_{i+1} + \pi/2$



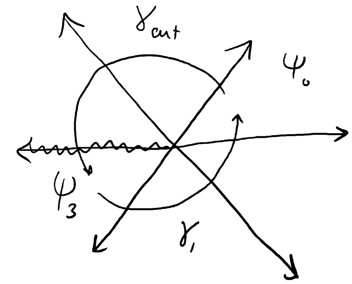
$Y_i(z)$ can be continued & $Y_i(z) \sim Y(z)$ remains true

Stokes Matrix

- assume $\ln(z)$ is defined by making a cut along d_e.

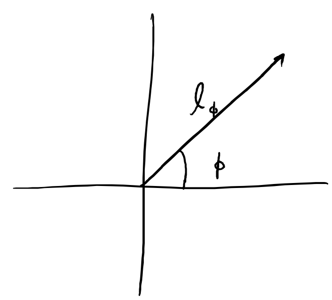
$$\Psi_3 = \Psi_0 \, S_- \quad \text{along } \gamma_1$$

$$\Psi_0 = \Psi_3 \, S_+ \, e^{2\pi i X^\circ} \quad \text{along } \gamma_{cut}$$



(S_-, S_+) are Stokes matrices.

$$\beta(z) = \sum_{n=1}^{\infty} z^n (n-1)!$$



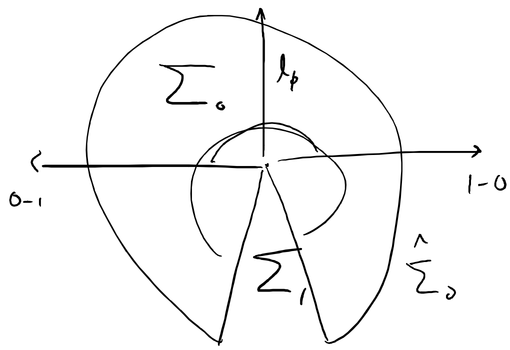
$$\int_{l_\phi} \frac{e^{-p/z}}{1-p} \, dp$$

$$= \left[\frac{1}{1-p} \, \frac{e^{-p/z}}{-z^{-1}} \right]_{l_\phi} - \int_{l_\phi} \frac{1}{(1-p)^2} \, \frac{e^{-p/z}}{-z^{-1}} \, dp$$

$$= \lim_{\substack{p \to \infty \\ \text{along } l_\phi}} \left(\frac{-e^{-pz^{-1}}}{(1-p)z^{-1}} \right) + z + z \int_{l_\phi} \frac{1}{(1-p)^2} \, e^{-p/z} \, dp$$

ℓ_ϕ

$$\sim \sum_{n=1}^{\infty} z^n (n-1)! \quad \text{as } z \to 0 \text{ along } \arg(z) \in \left(\phi - \tfrac{\pi}{2}, \phi + \tfrac{\pi}{2}\right)$$



O. Costin

Asymptotics & Summability pp 108.