ODE's over $\mathbb{C}$

$D \subset \mathbb{C}$ disc centered at 0.

$$A : D \setminus \{0\} \longrightarrow M_{N\times N}(\mathbb{C})$$

$$\boxed{F'(z) = A(z)\, F(z)}$$

Assume A does not have essential singularity at 0:

$$A(z) = \sum_{k=-r-1}^{\infty} A_k z^k \qquad \text{Laurent series.}$$

$\underline{r=-1}$ (i.e. A is holomorphic near 0)

(i.e. 0 is an ordinary point).

- $\exists!$ $F(z) = 1 + \sum_{k\geq 1} F_k z^k$ formal sol^n.
- radius of convergence of F = that of A.

$\underline{r=0}$ (i.e. A has a simple pole)

(i.e. 0 is a regular singular point)

- $\exists!$ solution $F(z)$ of the form

$$H(z) \cdot z^{A_{-1}} \qquad (H(z) = 1 + O(z))$$

(assumption: eigenvalues of A_{-1} do not differ by $\mathbb{Z}_{\neq 0}$)

- radius of convergence of H = that of A.

<u>$r \geq 1$</u> 0 is called <u>irregular singular point</u> of Poincare rank r.

<u>Drinfeld ODE</u>:

$$F'(z) = \left(\frac{A}{z} + \frac{B}{z-1} \right) F(z)$$

$$(A, B \in M_{N \times N}(\mathbb{C}))$$

Sol^{n}s: $F^{(0)}(z) = H^{(0)}(z)\, z^{A_{-1}}$

$F^{(1)}(z) = H^{(1)}(1-z)\, (1-z)^{B_{-1}}$

.

then $\forall z$ s.t. $|z|<1$ & $|1-z|<1$,

$$\boxed{F^{(0)}(z) = F^{(1)}(z) \cdot K}$$

1. Change of variables

$$w = z^{-1}, \quad dw = -z^{-2}\, dz$$

$$\frac{d}{dz} F(z) = A(z)\, F(z) \rightsquigarrow \frac{dF}{dw} = \frac{-1}{w^2} A\left(\tfrac{1}{w}\right) F$$

behavior at $z=\infty$ by defn is behavior at $w=0$.

eg

$$\frac{dF}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right) F$$

$$\underset{w=\frac{1}{z}}{\rightsquigarrow} \quad \frac{dF}{dw} = \frac{-1}{w^2}\left(Aw + \frac{Bw}{1-w}\right) F$$

$$= \left(-\frac{A}{w} - B\left(\frac{1}{w} + \frac{1}{1-w}\right)\right) F$$

$$= \left(\frac{-A-B}{w} + \frac{B}{w-1}\right) F$$

Summary: Drinfeld ODE has 3 regular

singularities $0, 1, \infty$ w/ residues

$A,\ B,\ -A-B.$

<u>Remark</u> we can consider A, B formal non-commutative variables

$$K(A,B) \in \mathbb{C}\langle\langle A, B\rangle\rangle$$

ring of formal power series in 2 non-comm variables

$$K = 1 \text{ if } \underset{\underset{AB-BA}{\parallel}}{[A,B]} = 0 \qquad \text{because } \underset{\underset{F^{(0)}}{\parallel}}{z^A}\ \underset{\underset{F^{(1)}}{\parallel}}{(1-z)^B}$$

" $K(A,B) = 1$ + terms involving commutators

$[A,B],\ [A,[A,B]],\ \ldots$ "

$$\frac{dF}{dz} = \left(\frac{A}{z} + \frac{B}{z-1}\right)F$$

<u>Smallest nontrivial example</u>

$$A = \begin{bmatrix} \frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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Solution near 0:

$$\begin{bmatrix} 1 & f(z;\lambda) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^{\lambda/2} & 0 \\ 0 & z^{-\lambda/2} \end{bmatrix}$$

$$f(z;\lambda) = \sum_{n=1}^{\infty} \frac{z^n}{\lambda - n}$$

$$= \ln(1-z) - \sum_{\ell \geq 1} \lambda^{\ell} \left(\sum_{n=1}^{\infty} \frac{z^n}{n^{\ell+1}} \right)$$

$$\rightsquigarrow K = \begin{bmatrix} 1 & \sum_{\ell \geq 1} \lambda^{\ell} \zeta(\ell+1) \\ 0 & 1 \end{bmatrix}$$

$$\left((1-z)^B = \exp(B \cdot \ln(1-z)) = \begin{bmatrix} 1 & \ln(1-z) \\ 0 & 1 \end{bmatrix} \right)$$

so we get cancellation.

More serious example $\quad (\lambda \notin \mathbb{Z})$

$$A = \begin{bmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}.$$

the differential eq^n is solved using hypergeometric fn.

Solution near 0 : $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} z^{\lambda/2} & 0 \\ 0 & z^{-\lambda/2} \end{bmatrix}$

Let

- $k = \sqrt{xy}$, $\quad r_1, r_2 = \dfrac{\lambda \pm \sqrt{\lambda^2 + 4xy}}{2}$

- $F(a,b;c;z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(a)_n (b)_n}{(c)_n}$

Gauss Hypergeometric Series

where $(p)_n = \begin{cases} p(p+1)\cdots(p+n-1) & ; \ n \geq 1 \\ 1 & ; \ n = 0 \end{cases}$

$$\alpha = (1-z)^{-k} F(r_1 - k, r_2 - k; \lambda; z)$$

$$\gamma = (1-z)^{-k} F(r_1 - k + 1, r_2 - k + 1; \lambda + 2; z) \left(\frac{-yz}{1+\lambda}\right)$$

$$\beta = (1-z)^k \; F(-r_1+k+1, -r_2+k+1; 2-\lambda; z) \left(\frac{-\chi z}{1-\lambda}\right)$$

$$\delta = (1-z)^k \; F(-r_1+k, -r_2+k; -\lambda; z)$$

$$(1-z)^{-t-1} = \sum_{\ell \geq 0} \binom{t+\ell}{\ell} z^\ell$$

$$\downarrow$$

$$\frac{(t+\ell)(t+\ell-1)\cdots(t+1)}{\ell!}$$

Hypergeometric Series $\quad (c \notin \mathbb{Z}_{\leq 0})$

$$\underset{2,1}{F}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(a)_n (b)_n}{(c)_n}$$

radius of convergence $= 1$

$\quad$ " $\quad = \infty$ if a or $b \in \mathbb{Z}_{\leq 0}$.

- $\dfrac{d}{dz} F(a, b; c; z) = \dfrac{ab}{c} F(a+1, b+1; c+1; z)$

- Hypergeometric eq$^{\underline{n}}$ $\quad$ (Gauss ~ 1813)

$$z(1-z)\frac{d^2F}{dz^2} + \left(c-(a+b+1)z\right)\frac{dF}{dz} - abF = 0.$$

- $F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$

(if $c-a-b \notin \mathbb{Z}$, z s.t. $|z|<1$ & $|1-z|<1$)

- $F(a,b;c;z) =$

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \cdot F(a,b;1-(c-a-b);1-z)$$

$$+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(-c+a+b)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; 1+(c-a-b); 1-z)$$

(here Γ is euler gamma fn)

(Barnes 1908)

(Whitaker-Watson Chapter 14)

Γ function (WW chapter 12) (Euler 1729)

- $\Gamma(x)$ is a meromorphic fn of $x \in \mathbb{C}$,

  the poles are all simple, at $x \in \mathbb{Z}_{\leq 0}$.

- $\Gamma(x+1) = x\,\Gamma(x)$

- Weierstrass:

$$\frac{1}{\Gamma(x)} = x \cdot e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n}$$

where $\gamma = \lim_{m \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{m} - \ln(m) \right)$

(Euler-Mascheroni Constant)

Euler:

$$\Gamma(x)\,\Gamma(1-x) = \frac{2\pi i}{e^{\pi i x} - e^{-\pi i x}}$$