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Lie Algebras & Representations

Defn - A lie algebra (over $\mathbb{C}$) is a $\mathbb{C}$-v.s. $\mathfrak{g}$ together with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$

s.t. (1) $[x, y] = -[y, z]$.

(2) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$.

(Jacobi identity)

Ex (1) V : vector space over $\mathbb{C}$.

$$\operatorname{End} V = \{ X : V \to V \text{ linear} \}$$

has the structure of a lie alg:

$$[A, B] := AB - BA$$

(2) $\mathfrak{g}$ any v.s., $[\cdot,\cdot] \equiv 0$ (abelian Lie alg).

A homomorphism of lie algebras $f : \mathfrak{g} \to \mathfrak{g}'$ is a linear map s.t. $f([x,y]) = [f(x), f(y)]$.

A representation of $\mathfrak{g}$ is a v.s. V over $\mathbb{C}$ together w/ a linear map $\mathfrak{g} \xrightarrow{\pi} \operatorname{End} V$ s.t.

$$\pi([x,y]) = \pi(x)\pi(y) - \pi(y)\pi(x)$$

i.e. a hom into $\mathfrak{gl}(V)$ $\qquad (\mathfrak{g} \circlearrowright V)$

<u>Notation</u>: $\mathfrak{gl}(V) = (\operatorname{End} V, \overset{\text{commutator}}{[\cdot,\cdot]})$. (Example 1).

<u>Ex</u> Let $\mathfrak{g}$ be any lie algebra.

For $x \in \mathfrak{g}$ we have a linear map

$$\overset{\text{adjoint}}{\operatorname{ad}(x)} : \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ y & \longmapsto & [x,y] \end{array}$$

$$\underbrace{\operatorname{ad} : \mathfrak{g} \longrightarrow \operatorname{End}(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g}) \text{ is a representation}}_{\text{this gives Jacobi id.}} \text{ (by Jacobi).}$$

<u>Operations on rep^{n}s</u>: Let $\mathfrak{g}$ be a lie algebra.

V_1, V_2 : two repns of $\mathfrak{g}$

$\mathfrak{g} \circlearrowright V_1, V_2$ $\Rightarrow$ $\mathfrak{g} \circlearrowright V_1 \oplus V_2$ componentwise

$\Downarrow$

$\mathfrak{g} \circlearrowright V_1 \otimes V_2$ by $X \cdot (v_1 \otimes v_2) = (X \cdot v_1) \otimes v_2 + v_1 \otimes (X \cdot v_2)$.

$\mathfrak{g} \circlearrowright V \rightsquigarrow \mathfrak{g} \circlearrowright V^*$ by $(x \cdot \xi)(v) = -\xi(x \cdot v)$

$\mathfrak{g} \circlearrowright \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ by $(x \cdot A)(v) = x \cdot (A(v)) - A(x \cdot v)$.

Remark

$$V^* \otimes W \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, W)$$
$$\xi \otimes w \longmapsto \{ v \longmapsto \xi(v) \cdot w \}$$

is a linear map which commutes w/ $\mathfrak{g}$-action

$\mathfrak{g}$-intertwiner: for V_1, V_2 two repns of $\mathfrak{g}$, a linear map $X : V_1 \longrightarrow V_2$ is a $\mathfrak{g}$-intertwiner if $X(x \cdot v) = x \cdot (X(v))$.

$\mathfrak{g} \curvearrowright V \rightsquigarrow V^{\mathfrak{g}} := \{v \in V \mid x \cdot v = 0 \ \forall x \in \mathfrak{g}\}$

space of $\mathfrak{g}$-invariant vectors

$$\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)^{\mathfrak{g}} = \mathfrak{g}\text{-intertwiners } V_1 \to V_2$$

Let V be a nonzero $\mathfrak{g}$-representation. We say

V is irreducible if

$$V' \subseteq V \text{ s.t. } \underset{\forall x \in \mathfrak{g}}{x \cdot V' \subseteq V'} \implies V' = 0 \text{ or } V.$$

V is called indecomposable if

$$V = V_1 \oplus V_2 \implies V_1 = 0 \text{ or } V_2 = 0$$

Schur's Lemma:

(1) Let $\mathfrak{g} \curvearrowright V_1, V_2$ be two irreducible representations, and let $X : V_1 \to V_2$ be a $\mathfrak{g}$-intertwiner. Then $X \equiv 0$ or an Isomorphism.

[Pf $\mathrm{Ker}(X) \subseteq V_1$ is a sub repn, so $\mathrm{Ker}\, X = 0$ or $\underbrace{\mathrm{Ker}\, X = V_1}$.

$X = 0$

$\mathrm{Ker}(X) = 0 \Rightarrow \mathrm{Im}(X) \subseteq V_2$ is a nonzero subrepn, so

$\mathrm{Im}(X) = V_2 \Rightarrow X$ is surj.]

(2) If $\mathfrak{g} \circlearrowright V$ is a f.d. irreducible repn

and $f : V \longrightarrow V$ is a $\mathfrak{g}$-intertwiner,

Then $\exists \lambda \in \mathbb{C}$ s.t. $f = \lambda \cdot id_V$.

need f.d., $\mathbb{C}$ alg. closed

[~~pf~~ Let $0 \neq v \in V$ be an eigenvector of f, with eigenvalue λ.

$$f(v) = \lambda v$$

$\mathrm{Ker}(f - \lambda \cdot id_V) \leq V$ is nonzero, so it's V.]

Example of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

$\mathfrak{sl}_2(\mathbb{C})$ = Lie alg of 2×2 matrices w/ trace 0.

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

3-dimensional w/ basis $\{h, e, f\}$

with commutations

$$[h, e] = 2e$$

$$[h, f] = -2f$$

$$[e, f] = h$$

<u>sl_2 - representations</u> $\equiv$ vector space V & 3 linear maps $\pi(h), \pi(e), \pi(f) \in \mathrm{End}(V)$ satisfying the 3 commutation relⁿs.

<u>e.g.</u> $sl_2 \curvearrowright \mathbb{C}^2$ naturally.

Irreducible repns of $sl_2(\mathbb{C})$

For every $n \in \mathbb{Z}_{\geq 0}$, consider $(n+1)$-dim'l vector space (L_n) with basis $\{m_0, \ldots, m_n\}$ and

$$\left.\begin{aligned} h \cdot m_j &= (n-2j)\, m_j \\ f \cdot m_j &= (j+1)\, m_{j+1} \\ e \cdot m_j &= (n-j+1)\, m_{j-1} \end{aligned}\right\} \quad m_{-1} = m_{n+1} = 0$$

$$\begin{tikzcd} 0 & m_0 \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] & m_1 \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] & \cdots \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] & m_{n-1} \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] & m_n \arrow[l, bend left, "e"] \arrow[r, bend left, "f"] & 0 \end{tikzcd}$$

$$h \quad [\, n \quad n-2 \quad n-4 \quad \cdots \quad -n+2 \quad -n \,]$$

<u>Ex</u>: L_n is irreducible

<u>Thm</u> Let V be an irreducible f.d. repⁿ of sl_2.

Let $n+1 = \dim(V)$. Then $V \cong L_n$.

<u>Proof</u> Let $0 \neq v \in V$ be an eigenvector for h,

let $\lambda \in \mathbb{C}$ be its eigenvalue ($h \cdot v = \lambda v$).

$$[h, e] = 2e \qquad \rightsquigarrow \quad e^k \cdot v \text{ is an eigenvector for } h$$
$$he = e(h+2) \qquad \text{w/ eigenvalue } \lambda + 2k$$

These are linearly indp (all different eigenvalues),

and V : f.d. $\Rightarrow$ $\exists k \in \mathbb{Z}_{\geq 0}$ s.t. $e^k \cdot v \neq 0$, $e^{k+1} \cdot v = 0$.

Let $v_0 := e^k \cdot v$, $\mu = \lambda + 2k$. $\quad e \cdot v_0 = 0$, $h \cdot v_0 = \mu v_0$.

$$v_\ell := \frac{f^\ell}{\ell!} \cdot v_0 \qquad \forall \ell \geq 0.$$

$$[h, f] = -2f \qquad \text{so} \qquad h \cdot v_\ell = (\mu - 2\ell) v_\ell$$
$$hf = f(h-2) \qquad\qquad f \cdot v_\ell = (\ell+1) v_{\ell+1}$$

<u>Claim</u>: $e \cdot v_\ell = (\mu - \ell + 1) v_{\ell-1} \quad (\forall \ell \geq 1)$

(~~pf~~ $\ell = 1$ $\quad e \cdot v_1 = e \cdot f(v_0) = \underbrace{\cancel{fe(v_0)}}_{0} + h(v_0) = \mu \cdot v_0$

$\ell > 1$ $\quad e \cdot v_\ell = \frac{(ef)}{\ell} \underbrace{\frac{(f^{\ell-1} \cdot v)}{(\ell-1)!}}_{v_{\ell-1}} = \frac{fe}{\ell}(v_{\ell-1}) + \frac{h}{\ell}(v_{\ell-1})$

$$= \frac{1}{\ell} f((\mu - \ell + 2) v_{\ell-2}) + \frac{1}{\ell}(\mu - 2(\ell-1)) v_{\ell-1}$$

$$= \frac{1}{l}\Big((\mu - l + 2)(l-1) + \mu - 2l + 2\Big)\, v_{l-1}$$

$$= \frac{1}{l}\big(l\cdot\mu - l(l-1)\big)v_{l-1} = (\mu - l + 1)v_{l-1}\ .\Big)$$

Let $p \in \mathbb{Z}_{\geq 0}$ be s.t. $v_p \neq 0$ but $v_{p+1} = 0$.

$$v_{p+1} = 0 \Longrightarrow e\, v_{p+1} = 0$$

$$\Downarrow$$

$$(\mu - (p+1) + 1)\, v_p = 0$$

$$\Downarrow$$

$$\mu = p \in \mathbb{Z}_{\geq 0}$$

$\underbrace{\operatorname{span}\{v_0, \dots, v_p\} \subseteq V}_{\text{subrepn}} \Longrightarrow V = \operatorname{span}\{v_0, \dots, v_p\}$ by irred. $\square$