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Friday, September 27, 2019 13:52

De Concini - Procesi Associators

D = (Dynkin) diagram of our root system.

For every $\underset{\text{connected}}{B \subseteq D}$; $\quad x_B = \sum_{i \in B} \alpha_i$

$\mathrm{Mns}(D)$ = max'l nested sets in D.

(1) $\forall\, \mathcal{S} \in \mathrm{Mns}(D)$, $B \in \mathcal{S}$,

$\{x_{B'} \,.\, B' \subset B \,;\, B' \in \mathcal{S}\}$ is a basis of $\mathfrak{h}_B^*$.

(2) $$\rho_{\mathcal{S}} : \mathbb{C}^{\mathcal{S}} \longrightarrow \mathfrak{h}$$

$$(\underline{u}) \longmapsto \Big(x_B = \prod_{\substack{C \in \mathcal{S} \\ B \subseteq C}} u_C \Big)$$

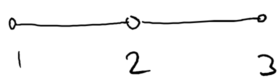
(3) $\forall\, \alpha \in R_+$, $\exists$ a polynomial $P_\alpha(\underline{u})$

(B_α = minimal element of $\mathcal{S}$ s.t. $\alpha \in \mathfrak{h}_{B_\alpha}^*$)

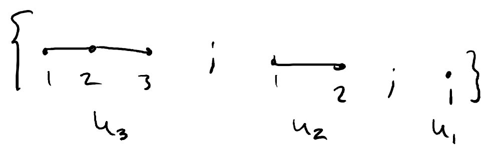
- $P_\alpha(\underline{u})$ depends on $u_{B'}$, $B' \in \mathcal{S}$, $B' \subsetneq B_\alpha$.

- $P_\alpha(\underline{0}) = 1$

- $$\alpha \sim \prod_{\substack{C \in \mathcal{S} \\ B_\alpha \subset C}} u_C \cdot P_\alpha(\underline{u})$$

e.g. 

: D

$\mathcal{S}$: 

$$\mathbb{C}^3 \longrightarrow \mathfrak{g}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = u_3$$
$$\alpha_1 + \alpha_2 = u_2 u_3$$
$$\alpha_1 = u_1 u_2 u_3$$

$$\alpha_2 = u_2 u_3 \underbrace{(1 - u_1)}_{P_{\alpha_2}}$$

$$\alpha_2 + \alpha_3 = u_3(1 - u_1 u_2)$$

$$\underbrace{}_{P_{\alpha_2+\alpha_3}}$$

Cor

(1) $$\mathbb{C}^S \xrightarrow{\rho_S} \mathfrak{h} \supset H_\alpha = \mathrm{Ker}(\alpha)$$

$$\rho_S^{-1}(H_\alpha) = \{u_C = 0\}_{\substack{C \in S \\ B_\alpha \subset C}} \cup \{P_\alpha = 0\}$$

$$\frac{d\alpha}{\alpha} = \sum_{\substack{C \in S \\ B_\alpha \subset C}} \frac{du_C}{u_C} + \frac{dP_\alpha}{P_\alpha}$$

$\shortparallel$

$d(\log \alpha)$

$$\nabla = d - \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} t_\alpha$$

$$\Rightarrow \quad \nabla = d - \sum_{\alpha \in R_+} \Bigg(\sum_{\substack{C \in S \\ B_\alpha \subset C}} \frac{du_C}{C} + \frac{dP_\alpha}{P_\alpha} \Bigg) t_\alpha$$

$$= d - \sum_{B \in S} \frac{du_B}{u_B} \underbrace{\Bigg(\sum_{\alpha \in \mathfrak{h}_B^* \cap R_+} t_\alpha \Bigg)}_{\substack{t_B \\ (\text{defn})}} - \underbrace{\sum_{\alpha \in R_+} \frac{dP_\alpha}{P_\alpha} t_\alpha}_{\text{Regular near } \underline{0}}$$

<u>Exercise</u>: Holonomy $\mathrm{rel}^{\underline{n}}$s for $\{t_\sim\}$

Exercise: Holonomy rel^{n}s for $\{t_\alpha\}_{\alpha \in R_+}$

$$\Rightarrow [t_{B_1}, t_{B_2}] = 0 \quad \forall B_1, B_2 \in S.$$

this is one of the "normal crossing type"

$\Rightarrow$ we have a unique sol^{n} of $\nabla \Psi = 0$ of the following form.

$$\Psi = H(\underline{u}) \cdot \prod_{B \in S} u_B^{t_B}$$

holomorphic near $\underline{0} \in \mathbb{C}^S$
$H(0) = 1$.

Holonomy Rel^{n}s:

$Y \subset R_+$ max'l s.t. Span$(Y) = 2d$,

$$\left[\sum_{\alpha \in Y} t_\alpha, t_\beta\right] = 0 \quad \forall \beta \in Y.$$

$$\prod_{B \in S} u_B^{t_B} = \prod_{B \in S} x_B^{r_B}$$

κ $\quad$ $\{B_1, \dots, B_k\}$ max'l elts

$$r_B = t_B - \sum_{i=1}^{k} t_{B_i}$$

$\{B_1, \dots, B_k\}$ max'l elts of $S|_B = \{B' \in S \mid B' \subsetneq B\}$

$$x_B = \prod_{\substack{C \in S \\ C \geq B}} u_C \longleftrightarrow u_B = \begin{cases} x_B & \text{if } B = D \\ \dfrac{x_B}{x_{C(B)}} & \text{o/w} \end{cases}$$

smallest in S s.t. $C(B) \supsetneq B$.

$$x^r = \exp(r \ln x)$$

On $\mathcal{C}^\circ := \{h \in \mathfrak{h} \mid \alpha_i(h) \in \mathbb{R}_{>0} \ \forall i\}$

for any max'l nested set S, we have a single-valued soln of $\nabla \Psi = 0$ on $\mathcal{C}^\circ$.

$\forall \mathcal{F}, \mathcal{G} \in Mns(D)$,

ind. of y

$$\Phi_{\mathcal{G}\mathcal{F}} := (\Psi_{\mathcal{G}}(y))^{-1} \Psi_{\mathcal{F}}(y) \quad \text{for } y \in \mathcal{C}^\circ$$

DCP Associat

DCP Associator

<u>Ex</u>: DCP Associator for A_2 = Drinfeld associator

$$\left(\frac{dF}{dz} = \left(\frac{A}{z} + \frac{B}{z-1} \right) F \right)$$

with $A = t_1$, $B = -t_2$

$$R_+ = \{ \alpha_1 , \alpha_2 , \alpha_1 + \alpha_2 \}$$

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r] & \mathfrak{h} & \mathbb{C}^2 \arrow[l] \end{tikzcd}$$

$$(u_1, u_2) \longmapsto \begin{array}{l} \alpha_1 + \alpha_2 = u_2 \\ \alpha_1 = u_1 u_2 \end{array}$$

$$\begin{array}{l} \alpha_1 + \alpha_2 = v_2 \\ \alpha_2 = v_1 v_2 \end{array} \longleftarrow (v_1, v_2)$$

$$w = u_2 = v_2 = \alpha_1 + \alpha_2$$

$$u_1 + v_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2} + \frac{\alpha_2}{\alpha_1 + \alpha_2} = 1$$

$$u_1 = z, \quad v_1 = 1 - z$$

DCP associator = associator of $\frac{dF}{dz} = \left(\frac{t_1}{z} + \frac{t_2}{1-z} \right) F$

Drinfeld (Quazihopf alg 1990)

(in the context of KZ eq^{n}s where the hyperplane arrangement is of type A)

Cherednik (Monodromy of repns for generalized KZ-eqns, RIMS (1990))

DCP $\left\{ \begin{array}{l} \text{Wonderful models} \\ \text{Hyperplane arrangements} \quad \text{selecta 1995} \end{array} \right\}$

Geometric Side

- let $\mathrm{Irr}(R)$ = set of irreducible root subsystems of R.
  (connected subdiagrams, up to W-action)

- $A \in \mathrm{Irr}(R)$; $A^\perp := \left\{ h \in \mathfrak{h} \;\middle|\; \begin{array}{c} \alpha(h)=0 \\ \forall \alpha \in A \end{array} \right\}$

$$\mathfrak{h} \setminus A^\perp = \mathbb{P}(\mathfrak{h}/A^\perp)$$

$$\forall A \longrightarrow \begin{array}{c} \cup \\ \mathfrak{h}^{reg} \end{array}$$

- Definition (Wonderful Model)

$$\mathfrak{h}^{reg} \longrightarrow \mathfrak{h} \times \prod_{A \in Irr(R)} \mathbb{P}_A$$

$$Y_R := \text{Closure of } \eta(\mathfrak{h}^{reg}) \subset \mathfrak{h} \times \prod_{A \in Irr(R)} \mathbb{P}_A$$

$$\left(\mathbb{P}_A = \mathbb{P}(\mathfrak{h}/A^{\perp}) \right)$$

$$\begin{tikzcd}
Y_R \arrow[r, "\pi"] & \mathfrak{h} \\
\tilde{\mathfrak{D}} = \pi^{-1}(\mathfrak{D}) \arrow[u, phantom, "\cup"] \arrow[r] & \mathfrak{D} = \bigcup_{\alpha \in R} H_\alpha \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

<u>Theorem</u> (i) π induces an isomorphism

$$Y_R \setminus \tilde{\mathfrak{D}} \longrightarrow \mathfrak{h} \setminus \mathfrak{D} = \mathfrak{h}^{reg}$$

(ii) $\mathfrak{D}$ is normal crossing

(iii) Irreducible components of $\tilde{\mathfrak{D}}$ are

$$\left\{ \mathfrak{D}_B = \pi^{-1}(B^{\perp}) \right\}_{B \in Irr(R)}$$

(iv) $\{\mathfrak{D}_B\}_{B \in \tau}$ intersects nontrivially iff τ is nested

(o) Y_R is smooth irr. variety

<u>Remark</u> maximal nested sets label asymptotics of approaching 0 from within $\mathcal{C}^\circ$.

S m.n.s. $\rightsquigarrow$ $\rho_S : \mathbb{C}^S \longrightarrow \mathfrak{h}$

$\rightsquigarrow$ $P_\alpha(\underline{u})$

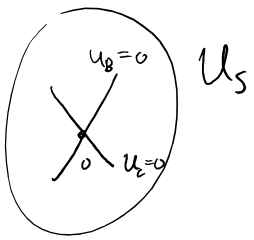
$$\mathcal{U}_S := \mathbb{C}^S \setminus \bigcup_{\alpha \in R} \{P_\alpha = 0\}$$

(v) $\{\mathcal{U}_S\}_{S \text{ m.n.s}}$ give an open covering of Y_R.

(vi) If $B \in \mathrm{Irr}(R)$, then

$$\mathcal{D}_B \cap \mathcal{U}_S \neq \emptyset \iff B \in S$$

in which case, $\mathcal{D}_B \cap \mathcal{U}_S = \{u_B = 0\}$


$u_B=0$
$u_c=0$
0
U_5