9/25

Wednesday, September 25, 2019 13:51

R - root system.

$\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C}$

$\nabla = d - \sum_{\alpha \in R} \frac{d\alpha}{\alpha} t_\alpha$, $t_\alpha \in End_{\mathbb{C}}(F)$, $t_{-\alpha} = t_\alpha$.

Holonomy relns: $\forall\, Y \subseteq R_+$ max'l s.t. $\mathbb{C}$-span(Y) is 2d,

we have $\left[\sum_{\alpha \in Y} t_\alpha \;\middle|\; t_\beta \right] = 0 \quad \forall\, \beta \in Y$.

W-equivariance: (assuming $W \circlearrowright F$)

$w t_\alpha w^{-1} = t_{w(\alpha)}$

D: (Dynkin) diagram of a root system, connected.

$B_1, B_2 \subset D$ subdiagrams (full subgraphs).

$B_1 \perp B_2$ if $\begin{cases} B_1, B_2 \text{ have no common vertices,} \\ \forall \alpha \in B_1, \beta \in B_2, \quad (\alpha, \beta) \notin D. \end{cases}$

- B_1 & B_2 are compatible if $B_1 \subset B_2$, $B_2 \subset B_1$, or $B_1 \perp B_2$.

- Nested set in D: collection of pairwise compatible connected subdiagrams.

- Maximal Nested sets.

<u>Ex</u> A_n $\overset{1}{\circ} - \overset{2}{\circ} - \overset{3}{\circ} - \cdots - \overset{n-1}{\circ} - \overset{n}{\circ}$

(or B_n or C_n)

Connected Subdiagram: $i - (i+1) - \cdots - (j-1) - j$

Brackets on $n+1$ variables $x_1\ x_2\ \cdots\ x_{n+1}$

$$i - \cdots - j \longleftrightarrow x_1 \cdots x_{i-1}\,(x_i \cdots x_{j+1})\,x_{j+2} \cdots x_{n+1}$$

Max-l nested sets $\longleftrightarrow$ complete bracketings of $n+1$ variables.

A_3 $\quad 1 - 2 - 3$

$$F_1 = \{1\ ;\ 1 - 2\ ;\ 1 - 2 - 3\} \longleftrightarrow (((x_1 x_2)x_3)x_4)$$

$$\frac{1}{n+1}\binom{2n}{n} = n^{th} \text{ catalan } \#$$

$\|$

#M.N.S. of A_n.

Elementary Properties of MNS-s.

Let $\mathcal{F}$ be a max'l nested set; and $B \in \mathcal{F}$.

<u>Lemma</u> $\mathcal{F}\big|_B = \{ B' \in \mathcal{F} \mid B' \subsetneq B \}$.

Let $B_1, \ldots, B_k$ be max'l elements of $\mathcal{F}|_B$.

Then: (1) $B_i \perp B_j \quad \forall\, i \neq j$ (obvious)

(2) $\exists !$ vertex $\alpha \in B$ s.t. $\alpha \notin B_i \quad \forall i$.

i.e. $B_1, \ldots, B_k$ are the connected components of $B \setminus \alpha$.

In our case (D: Dynkin); $k = 1, 2,$ or 2.

Pf $\bigcup B_i = B$ contradicts (i)

$\left| B \setminus \bigcup B_i \right| \geq 2$ contradicts maximality of $\mathcal{F}$. $\square$

<u>Cor</u> $|\mathcal{F}| = \dim \mathfrak{h}^* = |D|$

$\underline{\text{Def}}$ Let $\{x_B\}_{\substack{B \subseteq D \\ \text{connected}}} \subset \mathfrak{h}^*$.

$\{x_B\}_{\substack{B \subseteq D \\ \text{connected}}}$ is called an <u>adapted family</u> if for every

maximal nested set $\mathcal{F}$; $B \in \mathcal{F}$, the set

$\{x_{B'}\}_{\substack{B' \in \mathcal{F} \\ B' \subseteq B}}$ is a basis of $\mathfrak{h}_B^* = \mathrm{Span}(B) \subset \mathfrak{h}^*$.

(in particular, $\{x_B\}_{B \in \mathcal{F}}$ is a basis of $\mathfrak{h}^*$).

According to the Lemma, all we are requiring is that

$$x_B \in \mathfrak{h}_B^* \setminus \left(\bigoplus_{i=1}^{k} \mathfrak{h}_{B_i}^* \right).$$

eg (1) $x_B = \sum_{\alpha \in B} \alpha$ gives an adapted family.

(2) $B \subset D \rightsquigarrow R_B = \mathfrak{h}_B^* \cap R$,

$x_B = \sum_{\alpha \in (R_B)_+} \alpha$ also works eg $\quad \left(D = \underset{1}{\bullet} - \underset{2}{\bullet} —— \underset{3}{\bullet} \right)$

$$\chi_{\underset{1}{\bullet}} = \alpha_1, \quad \chi_{\underset{2}{\bullet}} = \alpha_2, \quad \chi_{\underset{1}{\bullet}-\underset{2}{\bullet}} = \alpha_1 + \alpha_2 ,$$ et cetera.

$$\mathcal{F} = \left\{ \underset{1}{\bullet} \; ; \; \underset{1}{\bullet}-\underset{2}{\bullet} \; ; \; \underset{1}{\bullet}-\underset{2}{\bullet}-\underset{3}{\bullet} \right\}$$

α_1 $\qquad$ B

$\left\{ \chi_{\underset{1}{\bullet}} , \chi_{\underset{1}{\bullet}-\underset{2}{\bullet}} \right\}$ should be a basis of $\mathbb{C}\alpha_1 + \mathbb{C}\alpha_2$.

<u>Change of Coordinates</u> (depends on m.n.s. $\mathcal{F}$)

$$\begin{array}{ccc} \mathbb{C}^{\mathcal{F}} & \xrightarrow{\quad \rho_{\mathcal{F}} \quad} & \mathfrak{h}^* \\ \cup & & \cup \\ \left(\underline{u} = (u_B)_{B \in \mathcal{F}} \right) & \longmapsto & h \end{array}$$

(birational map)

is uniquely determined by $\chi_B(h) = \prod_{\substack{C \in \mathcal{F} \\ B \subseteq C}} u_C \quad (\forall B \in \mathcal{F})$.

$$x_B = \prod_{\substack{C \in \mathcal{F} \\ B \subseteq C}} u_C$$ can be inverted as

$$\begin{cases} x_D & \text{if } B = D \end{cases}$$

$$u_B = \begin{cases} x_D & \text{if } B = D \\ \dfrac{x_B}{x_{c(B)}} & \text{o/w} \end{cases}$$

where $c(B)$ = minimal element of $\mathcal{F}$ which properly contains B.

eg B_2 $\quad R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$

$\underset{1}{\circ} \Leftarrow \underset{2}{\circ}$ $\qquad$ Adapted family : $x_1 = \alpha_1$, $x_2 = \alpha_2$,

$$x_{1-2} = \alpha_1 + \alpha_2$$

$$\mathcal{F} = \{ \underset{1}{\bullet} \; ; \; \underset{1}{\bullet}\!\!-\!\!\underset{2}{\bullet} \}$$

$$\rightsquigarrow \quad \begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathfrak{h}^* \\ \cup & & \cup \\ (u_1, u_2) & \longmapsto & h \\ \| & & \\ u_{1-2} & & \end{array} \quad \text{s.t.} \quad \begin{array}{l} (\alpha_1+\alpha_2)(h) = u_2, \\ \alpha_1(h) = u_1 \cdot u_2 . \end{array}$$

$\alpha_1 = u_1 \cdot u_2$, $\qquad \alpha_1 + \alpha_2 = u_2$

$u_1 = \dfrac{\alpha_1}{\alpha_1+\alpha_2}$, $\qquad u_2 = \alpha_1 + \alpha_2$.

Pull-back of a root ($\mathcal{F}$ fixed M.N.S., $\{x_B\}_{\substack{B \subseteq D \\ \text{connected}}}$: adapted)

$\alpha \in R$, express $\mathbb{C}^{\mathcal{F}} \xrightarrow{\rho_{\mathcal{F}}} \mathfrak{h}^* \xrightarrow{\alpha} \mathbb{C}$ in $\underline{u}$ variables.

Pick $B \in \mathcal{F}$ minimal s.t. $\alpha \in R_B$.

$\{x_{B'}\}_{\substack{B' \in \mathcal{F} \\ B' \subseteq B}}$ is a basis of $\mathfrak{h}_B^*$,

we can write $\alpha = \sum_{\substack{B' \in \mathcal{F} \\ B' \subseteq B}} a_{B'} \cdot x_{B'}$,

$a_{B'} \in \mathcal{F}$, $a_B \neq 0$

$$\alpha = a_B \underbrace{x_B}_{\prod_{\substack{C \in \mathcal{F} \\ B \subseteq C}} u_C} \Big(1 + \sum_{\substack{B' \in \mathcal{F} \\ B' \subsetneq B}} \frac{a_{B'}}{a_B} \cdot \underbrace{\frac{x_{B'}}{x_B}}_{\prod_{\substack{C \in \mathcal{F} \\ B' \subseteq C \subsetneq B}} u_C} \Big)$$

Prop $\forall \alpha \in R$, $\exists$ a polynomial $P_\alpha(\underline{u})$ s.t.

- $\alpha = a_B \prod_{\substack{C \in \mathcal{F} \\ B \subseteq C}} u_C \cdot P_\alpha(\underline{u})$
- P_α depends only on $\{u_{\ldots}\}$

- P_α depends only on $\{u_{B'}\}_{\substack{B' \in \mathcal{F} \\ B' \subsetneq B}}$
- $P_\alpha(\underline{0}) = 1$

B_2 example $\quad (x_1 = \alpha_1, \; x_2 = \alpha_2, \; x_{1-2} = \alpha_1 + \alpha_2)$

$$\mathcal{F} = \{ \overset{\bullet}{1} \, ; \, \overset{\bullet}{1} - \overset{\bullet}{2} \} \rightsquigarrow \quad \begin{aligned} \alpha_1 &= u_1 \cdot u_2 \cdot \underbrace{1}_{P_{\alpha_1}} \\ \alpha_1 + \alpha_2 &= u_2 \cdot \underbrace{1}_{P_{\alpha_1 + \alpha_2}} \end{aligned}$$

$$2\alpha_1 + \alpha_2 = u_2 \underbrace{(1 + u_1)}_{P_{2\alpha_1 + \alpha_2}}$$

$$\alpha_2 = u_2 \underbrace{(1 - u_1)}_{P_{\alpha_2}}$$