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Monday, September 23, 2019 13:52

Summary so far

- ODE's $/\mathbb{C}$
- PDE's with logarithmic poles along hyperplanes.

$$\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x$$

examples

Lie alg ~ Root Systems

V : f-d $\mathbb{C}$-vs

$X \subset V^* \setminus \{0\}$ finite

$t_x \in \mathrm{End}(F)$ for a f-d $\mathbb{C}$-vs F.

Recall Kohno's Lemma:

$$\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x \quad \text{is flat iff}$$

$\forall \; Y \subseteq X$ max'l s.t. span(Y) is 2d,

$$\left[\sum_{y \in Y} t_y , t_z \right] = 0 \quad \forall z \in Y.$$

Root System

E : $\mathbb{R}$-v.s

Complexification

$$\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C}$$

E : $\mathbb{R}$ - v.s

$R \subset E^* \setminus \{0\}$

$\mathfrak{h} = E \underset{\mathbb{R}}{\otimes} \mathbb{C}$

$\alpha : \mathfrak{h} \longrightarrow \mathbb{C} \quad \forall \alpha \in R$

$X \rightsquigarrow R_+$ ①

$\{ t_\alpha = t_{-\alpha} \}$ ②

$t_\alpha \in \mathrm{End}(F) \quad \forall \alpha \in R.$

$$\nabla = d - \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} t_\alpha$$
$$= d - \frac{1}{2} \sum_{\alpha \in R} \frac{d\alpha}{\alpha} t_\alpha$$

Base $\quad \mathfrak{h} \setminus \bigcup_{\alpha \in R} H_\alpha =: \mathfrak{h}^{reg}$

(regular elements)

$W \curvearrowright \mathfrak{h}^{reg}$ freely

—

Assume F has a W-action (i.e. we are given a grp hom $W \xrightarrow[\rho]{} GL(F)$).

We say ∇ is <u>W-equivariant</u> if

$$\underbrace{\rho(w)\, t_\alpha\, \rho(w)^{-1} = t_{w(\alpha)}}_{\text{in } \mathrm{End}(F).} \quad \forall \alpha \in R, w \in W.$$

In this case, base is $\mathfrak{h}^{reg}/W$.

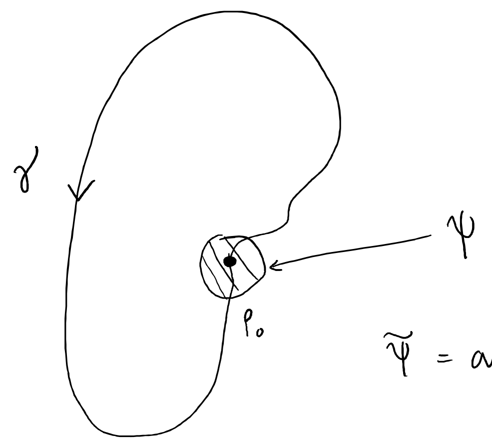
And we get the monodromy representation

$$\pi_1(\mathfrak{h}^{reg}/W \,;\, p_0) \xrightarrow{\mu} GL(F).$$

If $\Psi : \mathfrak{h}^{reg} \longrightarrow GL(F)$ is a soln of $\nabla \Psi = 0$, then so is $w \cdot \Psi$

$w \Psi(w^{-1} p)$

depends on p_0, depends on $GL(F)$-valued solution Ψ of $\nabla \Psi = 0$.



$\tilde{\Psi}$ = analytic continuation of Ψ along γ.

$$\boxed{\mu(\gamma) = \Psi^{-1}(p)\, \tilde{\Psi}(p)}$$

Brieskorn's Theorem

$$\pi_1\left(\mathfrak{h}^{reg}/W \,;\, p_0\right) = B_W$$

$\uparrow$ braid gp of type W

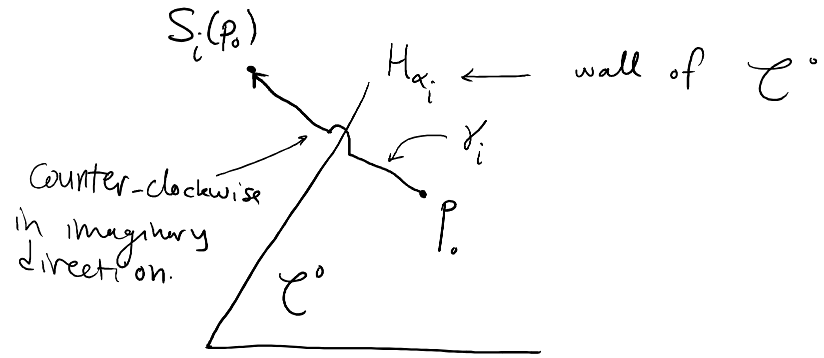
$$= \Big\langle T_i \ (i \in I) \ \Big| \ \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}} \Big\rangle$$

$$\pi_1\left(\mathfrak{h}^{reg}/W \,;\, p_0\right) \longleftarrow B_W$$

$$\begin{array}{ccc} \pi_1(\mathfrak{h}^{reg}/W ; p_0) & \longleftarrow & B_W \\ \Psi & & \Psi \\ \gamma_i & \longleftarrow & T_i \end{array}$$

$p_0 \in \mathcal{C}^0$ fundamental chamber

(i.e. $\alpha_i(p_0) \in \mathbb{R}_{>0}$)



Brieskorn: Die fundamentalgruppe des Raumes der regulären orbits einer endlichen Komplexen Spiegelungsgruppe (invent. math 1971 ; 57-61).

Deligne: les immeubles des groups de tresses généralisés

(Invent Math 1972)

theory of buildings.

Solutions of $\nabla \psi = 0$.

Recall $\quad F'(z) = \left(\frac{\Lambda}{z} + A_{reg}(z) \right) F(z)$

$\rightsquigarrow \exists!$ soln $\underset{\substack{\uparrow \\ \text{hol. near } 0.}}{H(z)} \cdot z^{\Lambda}$ $\quad$ w/ $H(0) = 1$

$$\rightsquigarrow \mu_{\psi}\left(\circlearrowleft_{0} \right) = e^{2\pi i \Lambda}$$

What is the several variables analogue.

Ex (Normal Crossing)

V is n-dim'l $\mathbb{C}$-v.s. $\quad \underset{\substack{\| \\ X}}{\{x_1, \dots, x_n\}}$ basis of V^*

Assume our PDE's are

$$\frac{\partial f}{\partial x_i} = \left(\frac{t_i}{x_i} + R_i(\underline{x}) \right) f \qquad (1 \le i \le n)$$

Consistent $\iff$ $[t_i, t_j] = 0 \quad \forall i, j$

$[t_i, R_j] = 0 \quad \forall i \neq j$

$$\frac{\partial R_i}{\partial x_j} - \frac{\partial R_j}{\partial x_i} + [R_i, R_j] = 0 \quad \forall i, j$$

In this case, $\exists!$ solution (w/ $H(\underline{0}) = 1$)

$$\underbrace{H(x_1, \dots x_n)}_{\text{hol. near } \underline{x} = \underline{0}} \underbrace{\prod_{i=1}^{n} x_i^{t_i}}_{\text{unambiguous since } t_i\text{'s commute}}$$

Ex (NOT normal crossing)

$$\nabla = d - \frac{dx}{x} t_1 - \frac{dy}{y} t_2 - \frac{d(x+y)}{x+y} t_3$$

(Kohno's lemm $t_1 + t_2 + t_3 =: T$ is central)

$\underline{Q}$ does $\exists$ a sol$_n$ of the form

$$H(x,y)\,\underline{x^{t_1} y^{t_2}}$$

$\underline{A}$ Probably not (if $[t_1, t_2] \neq 0$).

σ - process / blow up

$$\begin{aligned} x &= u \\ y &= uv \\ x+y &= u(1+v) \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} &\frac{dx}{x} = \frac{du}{u}, \quad \frac{dy}{y} = \frac{du}{u} + \frac{dv}{v} \\ &\frac{d(x+y)}{x+y} = \frac{du}{u} + \frac{dv}{1+v} \end{aligned}$$

$$\nabla = d - \frac{du}{u}(t_1 + t_2 + t_3) - \frac{dv}{v} t_2 - \frac{dv}{1+v} t_3$$

Now $\exists$ a soln of the form $H(u,v)\,\underbrace{u^T v^{t_2}}$

since $[T, t_2] = 0$.

Could swap role of x & y: $G(u,v)\, v^T u^{t_1}$

De Concini – Procesi: Hyperplane Arrangements & holonomy equations (Selecta 1995)

For any maximal nested set on the Dynkin diagram of R, there is a <u>fundamental solution</u>.

D = Dynkin diagram (Connectedness is the only relevant thing)

Let $B_1, B_2 \subset D$ be two full subdiagrams.

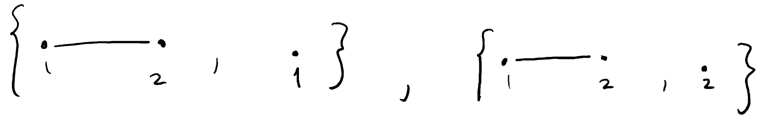
$B_1 \perp B_2$ means they don't share a vertex and $\forall \alpha \in B_1, \beta \in B_2$, α and β aren't connected in D.

B_1 & B_2 are compatible if either $B_1 \subset B_2$, $B_2 \subset B_1$, or $B_1 \perp B_2$.

Nested set = a set $\mathcal{H}$ of sub-diagrams of D s.t. B_1, B_2 $\mathcal{H}$ are compatible.

Maximal Nested set = Nested & Maximal (wrt inclusion).

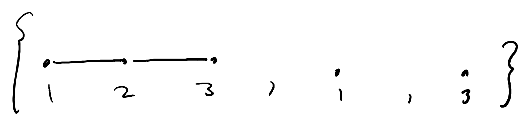
Rank 2:

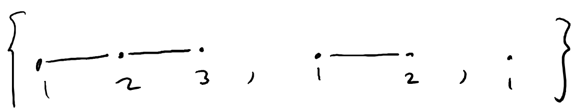


Rank 3:

are not compatible

There are 5 max'l nested sets here

 $\sim ((1\ 2)(3\ 4))$

 $\sim (((1\ 2)\ 3)\ 4)$

In type A, MNS $\longleftrightarrow$ Brackets on $\underbrace{x_1, x_2, \ldots, x_{n+1}}$

counted by Catalan #.