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(same notation as last time).

$W = \langle s_\alpha \rangle_{\alpha \in R}$ - Weyl group of root system.

<u>Properties of W</u>.

(1) $|W| < \infty$, preserves

$(\cdot, \cdot)$, $E^* \times E \to \mathbb{R}$, etc

(2) $W \circlearrowright \pi_0(E^\circ)$ is transitive

set of chambers

(3) (C° fund chamber)

simple reflections.

W generated by $\{ s_i = s_{\alpha_i} \}_{i \in I}$

simple roots $\longrightarrow \{\alpha_i\}_{i \in I}$ walls of C°.

(4) We have $\ell : W \longrightarrow \mathbb{Z}_{\geq 0}$ length fn:

$$\ell(w) = \min \{ k \mid \exists\, i_1, \dots, i_k \in I \text{ w/ } w = s_{i_1} \cdots s_{i_k} \}.$$

An expression $w = s_{i_1} \cdots s_{i_\ell}$ is reduced if $\ell = \ell(w)$.

(5) For $w \in W$ $i \in I$: TFAE

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$\ell(w) = 0 \Leftrightarrow w = e$
$\ell(w) = 1 \Leftrightarrow w = s_i$ for some i.

- $\ell(ws_i) < \ell(w)$
- $w(\alpha_i) \in R_-$
- $\forall$ reduced expression $w = s_{i_1} \cdots s_{i_\ell}$,
  $\exists\, j \in \{1, \dots, \ell\}$ s.t. $s_{i_j} s_{i_{j+1}} \cdots s_{i_\ell} = s_{i_{j+1}} \cdots s_{i_\ell} s_i$

exchange property.

(in particular, $w = s_{i_1} \cdots \hat{s}_{i_j} \cdots s_{i_\ell} s_i$ is reduced).

<u>Lemma</u> Let $a \in \mathcal{C}^\circ$ and $w \in W$ s.t. $w(a) \in \mathcal{C}^\circ$. Then $w = e$.

Pf Assume $w \neq e$ and $\ell(w) = \ell$.

Pick a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$.

$$\ell(ws_{i_\ell}) < \ell(w) \Rightarrow w(\alpha_{i_\ell}) \in R_-$$

$$\left.\begin{array}{l} \underbrace{\alpha_{i_\ell}(a)}_{\displaystyle \| \atop \displaystyle (w(\alpha_{i_\ell}))(w(a))} > 0 \end{array}\right\} \text{contradiction}$$

$\square$

<u>Remark</u> If $a \in \overline{\mathcal{C}^\circ}$ and $\overset{\text{stabilizer}}{W_a} = \{ w \in W \mid w(a) = a \}$

then $w = s_{i_1} \cdots s_{i_\ell} \in W_a \Leftrightarrow s_{i_j} \in W_a \;\; \forall\, j \in \{1, \dots, \ell\}$.

So: $W \circlearrowright \pi_0(E^\circ)$ free & transitive

$$\begin{array}{ccc} W & \xleftrightarrow{\text{bijection}} & \pi_0(E^\circ) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ w & \longmapsto & w(C^\circ) \end{array}$$

Recall rank 2 relations.

$\forall i, j \in I,\ i \neq j,\ (s_i s_j)^{m_{ij}} = e$ where

$a_{ij} a_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

<u>Theorem</u> W admits the following presentation

$$W = \Big\langle s_i\ (i \in I) \ \Big|\ \underset{\forall i}{s_i^2 = e},\ \underset{\forall i \neq j}{(s_i s_j)^{m_{ij}} = e} \Big\rangle$$

<u>Remark</u> In general, a group admitting a presentation as above is called a coxeter gp.

(So Weyl gp is a Coxeter gp)

(D_{2n} is a finite coxeter gp)

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<u>Lemma</u> Let M be a monoid and let $T_i \in M$ $(\forall i \in I)$ s.t.
(Jacques Tits)

$$\underbrace{T_i T_j T_i T_j \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}} \quad \text{(could be odd)}$$

For a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, the element $T_{i_1} T_{i_2} \cdots T_{i_\ell}$ depends only on w.

<u>Proof of Theorem</u> (assuming Lemma)

Let G be a gp, $f : I \longrightarrow G$ set map
$i \longmapsto f_i$

s.t. $f_i^2 \ \forall i$ and $(f_i f_j)^{m_{ij}} = e$,

<u>To prove</u> $\exists!$ gp-hom $g : W \longrightarrow G$ s.t. $g(s_i) = f_i$.

Define $g(w) \in G$ as:

pick a reduced exp $w = s_{i_1} \cdots s_{i_\ell}$, then

$$g(w) = f_{i_1} \cdots f_{i_\ell}.$$

Lemma says this is well-defined.

uniqueness follows since $\{S_i\}$ generates W.

<u>Check</u>: g is a gp hom.

E.T.S. $g(S_i w) = g(S_i) g(w)$.

<u>two cases</u>

(1) $\ell(S_i w) > \ell(w)$.

Then $S_i S_{i_1} \cdots S_{i_\ell}$ is reduced ✓

(2) $\ell(S_i w) < \ell(w)$

let $u = S_i w$. back to case 1!

$\ell(S_i u) > \ell(u)$.

$\square$

<u>Proof of Tits' lemma</u>

By induction on $\ell(w)$. Base case

$\ell(w) = 0$ (or $\ell(w) = 1$) is obvious.

Let $w = S_{i_1} \cdots S_{i_\ell} = S_{j_1} \cdots S_{j_\ell}$ be two reduced expressions,

Observation: if $i_1 = j_1$ or $i_\ell = j_\ell$, we are done by induction.

$(T_{i_1} \cdots T_{i_\ell} = T_{j_1} \cdots T_{j_\ell} \leftarrow$ to prove$)$.

Otherwise

$$\left.\begin{array}{l} w = s_{i_1} \cdots s_{i_\ell} \\ \ell(w s_{j_\ell}) < \ell(w) \end{array}\right\} \xrightarrow[\text{property}]{\text{exchange}} w = s_{i_1} \cdots \hat{s}_{i_t} \cdots s_{i_\ell} s_{j_\ell}.$$

$t \neq 1 \Rightarrow$ we are done.

$t = 1 \qquad w = s_{i_2} \cdots s_{i_\ell} s_{j_\ell}$

$$\left.\begin{array}{l} w = s_{j_1} \cdots s_{j_\ell} \\ \ell(w s_{i}) < \ell(w) \end{array}\right\} \rightsquigarrow w = s_{i_2} \cdots \hat{s}_{i_r} \cdots s_{i_\ell}$$

$r \neq 1 \Rightarrow$ we are done

if bad stuff keeps happening:

$$\begin{array}{ccc} & w & \\ s_{i_1} \cdots s_{i_\ell} & & s_{j_1} \cdots s_{j_\ell} \\ \downarrow & & \downarrow \\ s_{i_2} \cdots s_{i_\ell} s_{j_\ell} & & s_{j_2} \cdots s_{j_\ell} s_{i_\ell} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

$$\cdots \underbrace{s_{j_\ell} s_{i_\ell} s_{j_\ell}}_{m_{i_\ell, j_\ell}} = \cdots \underbrace{s_{i_\ell} s_{j_\ell} s_{i_\ell}}_{m_{i_\ell, j_\ell}}$$

good stuff happens. □

Remark preferred (intrinsic to W) Exchange property (for any coxeter gp) is alternate way to define thes

Def The braid group B_W (corresponding to W) is the group w/ the following presentation.

$$\left\langle T_i \ (i \in I) \;\middle|\; \underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}} \quad \forall i \neq j \in I \right\rangle$$

Ex (type A_n)

$\mathbb{R}^{n+1}$ with standard inner product

$\mathbb{R}^{n+1} \supset E =$ Kernel of the linear form

$$\underline{x} \longmapsto \sum x_i$$

$$E = \{ \underline{x} \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0 \}$$

$$R \subset E^*$$

$$\|$$

$$\{ \alpha_{ij} = \varepsilon_i - \varepsilon_j \}_{\substack{1 \leq i, j \leq n+1 \\ i \neq j}}$$

$\varepsilon_i(\underline{x}) = x_i$

restricted to E

$$H_{ij} \subset E$$

$$\|$$

$$\{ \underline{x} \in \mathbb{R}^{n+1} \mid \sum x_k = 0 \}$$

$$\left\{ \underline{x} \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} \sum x_k = 0 \\ x_i = x_j \end{array} \right\}$$

$$E^\circ = E \setminus \bigcup_{i \neq j} H_{ij}$$

all coords distinct.

$$\mathcal{C}^\circ = \{ \underline{x} \in E \mid x_1 > x_2 > \cdots > x_{n+1} \}$$

$$R_+ = \{ \alpha_{ij} \mid 1 \le i < j \le n+1 \}, \quad R_- = -R_+.$$

$$\cup$$

$$\{ \alpha_{i,i+1} \}_{1 \le i \le n} \quad \text{simple roots}$$

$$S_{i,i+1} = (i \;\; i+1) \text{ on } \underline{x}$$

$$\rightsquigarrow W = S_{n+1} \longleftrightarrow \pi_0(E^\circ)$$

$$\sigma \longmapsto \mathcal{C}_\sigma = \{ \underline{x} \mid x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n+1)} \}$$

B_{n+1} = Artin's Braid group on $n+1$ strands

Braid group

$$B_{n+1} = \pi_1 \left(\mathrm{Conf}_{n+1}(\mathbb{C}) \right)$$

Brieskorn's thm 1971

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$$B_W = \pi_1\left(\left(E \otimes_{\mathbb{R}} \mathbb{C} \right) \setminus \bigcup_{\alpha \in R} \ker(\alpha) \Big/ W \right)$$

Deligne (1972)