9/18

Wednesday, September 18, 2019 13:51

(Same notation as last time)

<u>Weyl group of root system R</u>

$W \leq GL(E^*)$ (or $GL(E)$)

Subgroup generated by $\{S_\alpha\}_{\alpha \in R}$

(1) $|W| < \infty$, since W preserves R & R spans E^*, so $W \leq \mathrm{Perm}(R)$

(2) W preserves $(\cdot,\cdot)$ on E (or E^*)

$$(w\phi, w\psi) = (\phi, \psi)$$

<u>Proof</u> $(S_\alpha \phi, S_\alpha \psi) = (\phi - \alpha(\phi)\alpha^\vee, \psi - \alpha(\psi)\alpha^\vee)$,

and $(\phi, \alpha^\vee) = \dfrac{2}{(\alpha,\alpha)} \alpha(\phi)$, $(\alpha^\vee, \alpha^\vee) = \dfrac{4}{(\alpha,\alpha)}$.

Expand & done.

(3) W preserves $\{H_\alpha\}_{\alpha \in R}$ hyperplane arrangement in E.

$\Rightarrow$ W acts on set of connected components of $E^\circ = E \setminus \bigcup_{\alpha \in R} H_\alpha$.

simple roots

$$\mathcal{C}^\circ \subset E^\circ \text{ fund. chamber} \rightsquigarrow \underbrace{\{\alpha_i\}_{i\in I}}^{\text{simple roots}} \text{ walls of } \mathcal{C}^\circ$$

$W' =$ subgroup of W generated by $S_i = S_{\alpha_i}$ $(i \in I)$.

<u>Lemma</u> Let $y \in E$. Then $\exists\, w \in W'$ s.t. $w(y) \in \overline{\mathcal{C}^\circ}$.

(i.e. α_i scores ≥ 0 on $w(y)$ $\forall i \in I$).

<u>proof</u> pick $a \in \mathcal{C}^\circ$. Consider $W'\cdot y \overset{\text{finite}}{=} \{w(y) \mid w \in W'\}$.

Choose $y_0 \in W'\cdot y$ st. $\text{distance}(y_0, a) \leq \text{distance}(y', a)$ $\forall\, y' \in W'\cdot y$.

<u>Claim</u> $\forall i \in I$, $\alpha_i(y_0) \geq 0$.

Pf $\text{distance}(a, y_0)^2 \leq \text{distance}(a, S_i(y_0))^2$

$$|a - y_0|^2 \leq |a - S_i(y_0)|^2$$

$$\cancel{|a|^2} + \cancel{|y_0|^2} - 2(a, y_0) \leq \cancel{|a|^2} + \cancel{|S_i(y_0)|^2} - 2(a, S_i(y_0))$$

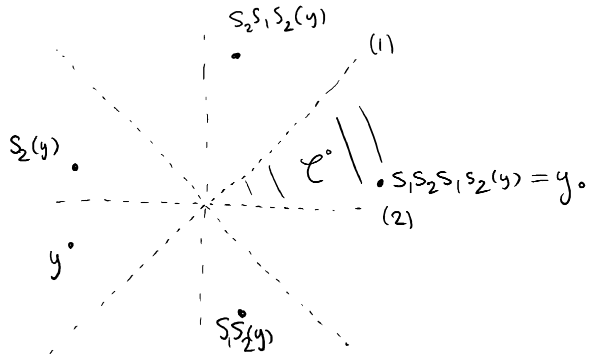
$$\Rightarrow \quad (a, y_0 - S_i(y_0)) \geq 0$$

$$(a, \alpha_i(y_0)\alpha_i^\vee) \geq 0$$

$$\alpha_i(y_0) \underbrace{(a, \alpha_i^\vee)}_{\text{positive since } a \in \mathcal{C}^\circ} \geq 0$$

$\square$

Ex



$$\forall y \in E, \quad \exists\, w \in W' = \langle s_i \rangle_{i \in I} \quad \text{s.t.} \quad w(y) \in \overline{\mathcal{C}^\circ}$$ } statement of thm above

Corollaries

(1) $W \circlearrowright \underset{\text{connected components}}{\pi_0}(E^\circ)$ is transitive

(2) $$R = \bigcup_{i \in I} W' \cdot \alpha_i$$

pf $\bigcup_{i \in I} W' \cdot \alpha_i \subset R$ obvious.

conversely, if $\alpha \in R$, pick $\underset{\text{chamber}}{\mathcal{C}} \subset E^\circ$ s.t. α is a wall of $\mathcal{C}$,

pick $w \in W'$ s.t. $w(\mathcal{C}) = \mathcal{C}^\circ$.

walls are $\{\alpha_i\}_{i \in I}$

$\Rightarrow$ $w(\alpha) = \alpha_i$ for som $i \in I$.

$$\boxed{s_i(\alpha_j) = \alpha_j - a_{ij}\, \alpha_i \qquad (i \neq j)}$$

$\{\alpha_i\}_{i \in I} \rightsquigarrow$ repeated application of $\{s_i\}_{i \in I}$ gives R.

(3) $W = W'$

pf $W = \langle S_\alpha \rangle_{\alpha \in R}$, WTS $S_\alpha \in W'$ $\forall\ \alpha \in R$.

let $w \in W'$ and $i \in I$ be s.t. $\alpha = w(\alpha_i)$

Then $S_\alpha = w \cdot S_i \cdot w^{-1} \in W'$ $\square$

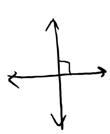
- generators of W : $\{S_i\}_{i \in I}$.
- Relations of W : $S_i^2 = e$ $\forall i \in I$.

Rank 2 relations (order of $s_i s_j = ?$)

$(|I| = \dim E)$

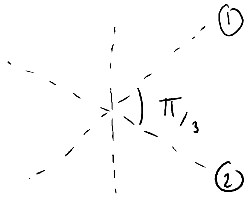
We use our classification of rank-2 root systems.

Eg



$S_1 S_2 = S_2 S_1$

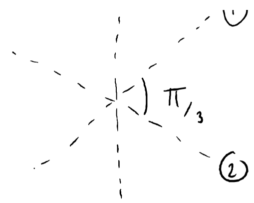
$A_1 \times A_1$



$S_1 S_2 =$ rotation by $\frac{2\pi}{3}$

$(S_1 S_2)^3 = e$

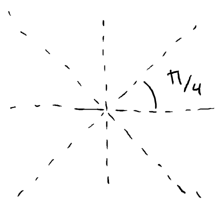
A_2



A_2

$s_1 s_2$ = rotation by $\frac{\quad}{3}$

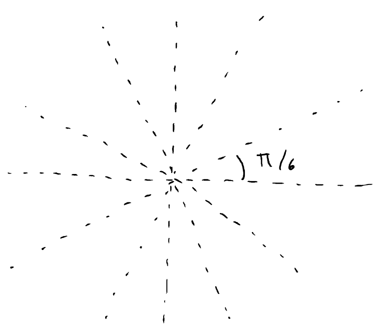
$$(s_1 s_2)^3 = e$$



B_2

$s_1 s_2$ = rotation by $\frac{\pi}{2}$

$$(s_1 s_2)^4 = e$$



G_2

$s_1 s_2$ = rotation by $\frac{2\pi}{6}$

$$(s_1 s_2)^6 = e$$

<u>Remark</u> D_{2n} comes from a root system

$$\Leftrightarrow \cos\left(\frac{2\pi}{n}\right) \in \mathbb{Q} \Leftrightarrow n = 2, 3, 4, 6$$

$\forall\, i \neq j$, $(s_i s_j)^{m_{ij}} = e$ where

$a_{ij}\, a_{ji}$	m_{ij}
0	2
1	3

0	2
1	3
2	4
3	6

<u>Definition</u> For $w \in W$, define length of w

$$l(w) = \min\{k \mid \exists\, i_1, \dots, i_k \in I \text{ s.t. } w = s_{i_1} \cdots s_{i_k}\}.$$

$w = s_{i_1} \cdots s_{i_l}$ is "reduced expression" of w if $l = l(w)$.

<u>Lemma</u> If $\alpha \in R_+$ and $s_i(\alpha) \in R_-$, then $\alpha = \alpha_i$.

Pf $s_i(\alpha) = \alpha - \alpha(\alpha_i^\vee) \cdot \alpha_i \in R_-$

if $\alpha = \sum_{j \in I} n_j \alpha_j$ then $n_j = 0 \ \forall j \neq i$

$\alpha \sim \alpha_i$, $\alpha \in R_+ \implies \alpha = \alpha_i$. $\square$

<u>Proposition</u> Let $w \in W$; $i \in I$. Then TFAE.

(1) $l(ws_i) < l(w)$

(2) $w(\alpha_i) \in R_-$

(3) For any reduced expression

$$w = s_{i_1} \cdots s_{i_\ell}$$

$\exists\, j \in \{1, \dots, \ell\}$ s.t. $s_{i_j} s_{i_{j+1}} \cdots s_{i_\ell} = s_{i_{j+1}} \cdots s_{i_\ell} s_i$ $\Bigg\}$ exchange property.

<u>Proof</u> <u>(2) $\Rightarrow$ (3)</u> $w(\alpha_i) \in R_-$, $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$.

Define $\beta_j = s_{i_{j+1}} \cdots s_{i_\ell} (\alpha_i)$ (for $0 \le j \le \ell$)

eg $\beta_0 = \underset{\substack{\Uparrow \\ R_-}}{w(\alpha_i)}$, $\beta_\ell = \underset{\substack{\Uparrow \\ R_+}}{\alpha_i}$.

$$\Rightarrow \exists\, j \in \{1, \dots, \ell\} \text{ s.t. } \beta_{j-1} \in R_-,\ \beta_j \in R_+$$

$$\beta_{j-1} = s_{i_j}(\beta_j) \Rightarrow \beta_j = \alpha_{i_j}$$

$$\text{i.e. } \alpha_{i_j} = \underbrace{s_{i_{j+1}} \cdots s_{i_\ell}}_{u} (\alpha_i)$$

$$\Rightarrow s_{i_j} = u \cdot s_i \cdot u^{-1}$$

$$\Rightarrow s_{i_j} u = u s_i .$$

<u>(3) $\Rightarrow$ (1)</u> Let $l = \ell(w)$ and $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced exp.

By 3, $w = s_{i_1} s_{i_2} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_\ell} s_i$

$$\Rightarrow w s_i = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_\ell} .$$

$$\Rightarrow \ell(w s_i) < \ell(w).$$

<u>(1) $\Rightarrow$ (2)</u> If not, then $w(\alpha_i) \in R_+$

$$\underbrace{w \cdot s_i}_{u} (\alpha_i) \in R_-$$

We have shown $u(\alpha_i) \in R_- \Rightarrow (3) \Rightarrow (1)$ for u

$\ell(u \cdot s_i) < \ell(u)$

$\ell(w) < \ell(w s_i)$

$\square$

Corollaries of proposition

(1) $W = \langle \underset{i \in I}{s_i} \mid s_i^2 = e, \ (s_i s_j)^{m_{ij}} = e \rangle$

(2) $W \circlearrowright \pi_0(E^\circ)$ is free (& transitive)

(3) Equivalent defns of $\ell(w)$:

$\ell(w) =$ smallest # of walls to cross to get from C° to $w(C^\circ)$.

$= \# \{ \alpha \in R_+ \mid w(\alpha) \in R_- \}$