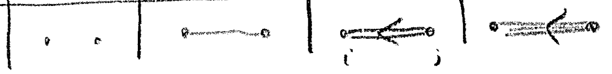
Recall: E : f-d $\mathbb{R}$-vs. $(\cdot,\cdot) : E^2 \to \mathbb{R}$ $E^* \xrightarrow[\nu]{\sim} E$ to get $(\cdot,\cdot)$ on E^*.

$R \subset E^* \setminus \{0\}$ root system if it satisfies a few properties. $=: E^\circ$

Pick $\mathcal{C}^\circ$, a connected component of $E \setminus \bigcup_{\alpha \in R} H_\alpha$, fundamental chamber.

$R_\pm = \{\alpha \in R \mid \pm\alpha(x) > 0 \ \forall x \in \mathcal{C}^\circ\}$. $A = \left(a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\right)_{i,j \in I}$, $a_{ij} \in \mathbb{Z}_{\le 0}$, $a_{ii} = 2$.

(a_{ij}, a_{ji})	$(0,0)$	$(-1,-1)$	$(-2,-1)$	$(-3,-1)$
Edges				

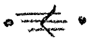


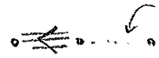
$\rightsquigarrow$ Γ = Dynkin diagram.

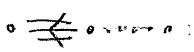
Assume Γ is connected (i.e. R is irreducible).

$\{\alpha_i\}_{i \in I}$ is a basis for E^*. if $d_i = \frac{(\alpha_i,\alpha_i)}{2} \in \mathbb{R}_{>0}$, then $[d_i a_{ij}]_{i,j \in I}$ = matrix of $(\cdot,\cdot)$ in some basis

If $\underline{u} = (u_i)_{i \in I} \in \mathbb{R}^I$ s.t. $\underline{u} \ge 0$ and $A\underline{u} \le 0$ then $\underline{u} = \underline{0}$. (bc. $(\underline{u},\underline{u}) = u^T A u$). $(*)$

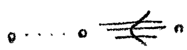
(1) if Γ contains a triple edge, then $\Gamma =$  (G_2).

pf Assume not. Then Γ contains  or .

 :

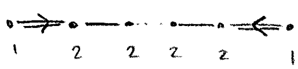
$(a, b > 0,\ ab \neq 0)$
$c, d \ge 0$

$$\begin{bmatrix} 2 & -3 & -c \\ -1 & 2 & -a \\ -d & -b & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -c \\ 1-a \\ 2-2b-3d \end{bmatrix}.$$ contradicts $(*)$.

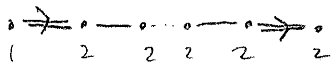


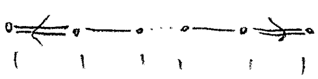
$$\begin{bmatrix} 2 & -a & -c \\ -b & 2 & -3 \\ -d & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-2a-c \\ 1-b \\ -d \end{bmatrix}.$$ also contradicts $(*)$.

(2) If Γ has a double-edge, it has only one



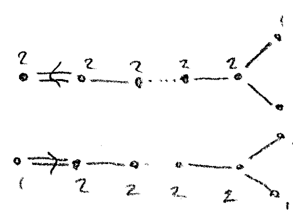
$\leftarrow$ $\underline{u}$ that contradicts $(*)$.

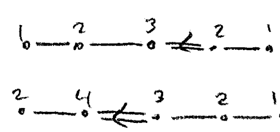


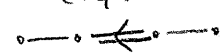


$$\begin{bmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & 0 \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

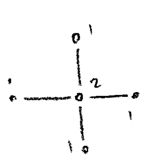
(3) If Γ has double-edge, valency of each vertex ≤ 2.

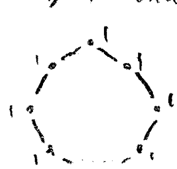


(4) 

so if Γ has a double edge, $\Gamma =$ , , or .

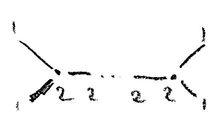
(5) From now on, Γ only has simple edges.





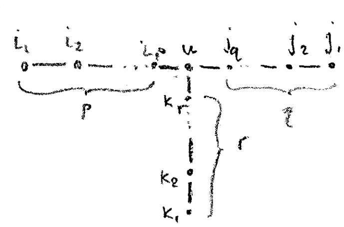
contradicts contradicts

So it's a tree where valence ≤ 3 everywhere.



Γ only has 0 or 1 vertices of valence 3.

Γ must look like:



$x = \sum_{t=1}^{p} t\,\alpha_{i_t}$ $z = \sum_{t=1}^{r} t\,\alpha_{k_t}$

$y = \sum_{t=1}^{q} t\,\alpha_{j_t}$ $w = \alpha_u$.

(1) x, y, z orthogonal

(2) $|x|^2 = p(p+1)$
$|y|^2 = q(q+1)$ $|w|^2 = 2$
$|z|^2 = r(r+1)$

3) $(x,w) = -p$, $(y,w) = -q$, $(z,w) = -r$.

Since $w \notin \mathrm{Span}(x,y,z)$, distance from w to space spanned by x, y, z > 0.

$$0 < (\text{distance})^2 = |w|^2 - \frac{(x,w)^2}{|x|^2} - \frac{(y,w)^2}{|y|^2} - \frac{(z,w)^2}{|z|^2} = 2 - \frac{p}{p+1} - \frac{q}{q+1} - \frac{r}{r+1}$$

$$= -1 + \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1}. \quad \text{So} \quad \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} > 1. \qquad \text{wolog } p \geq q \geq r.$$

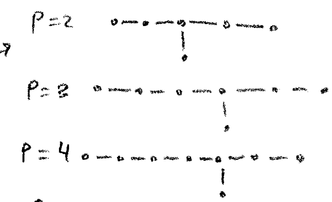
$$\text{so} \quad \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} \leq \frac{3}{r+1} \Rightarrow r+1 < 3 \Rightarrow r < 2 \text{ so } r = 0 \text{ or } 1.$$

if $r = 0$, we get  , (A_n).

if $r = 1$, $\frac{1}{p+1} + \frac{1}{q+1} > \frac{1}{2}$, so $\frac{2}{q+1} > \frac{1}{2} \Rightarrow q < 3$. So $q = 1$ or 2

if $q = 1$,  (D_n).

if $q = 2$, $\frac{1}{p+1} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, so $p < 5$. $p = 2, 3,$ or 4. This gives



$(E_6, E_7,$ and $E_8)$. ◻

<u>For existence</u> Bourbaki ch 4,5,6 (Lie gps & Lie alg) Table of root systems.

<u>Weyl Group of a root system.</u> W = group generated by $\{S_\alpha = \text{reflection thru } H_\alpha\}_{\alpha \in R}$,

It's a subgp of $GL(E^*)$. Note: W preserves R; (R spans E^*).

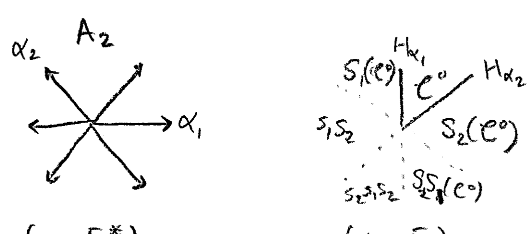
So $W \leq \mathrm{Permutations}(R)$ which is finite, so W is finite.

W acts on E and permutes H_α's $\Rightarrow$ W maps chambers to chambers.

So $W \curvearrowright \{\text{Connected components of } E^\circ\}$. This action is free & transitive.

in fact,
$$\begin{array}{ccc} W & \longleftrightarrow & \pi_0(E^\circ) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ w & \longleftrightarrow & w(\mathcal{C}^\circ) \end{array}$$

eg



$$W \cong S_3$$

(since $W = \langle S_1, S_2 \mid S_1^2 = S_2^2 = 1,\ S_2S_1S_2 = S_1S_2S_1 \rangle$).