9/13

Friday, September 13, 2019 13:52

Root systems

E : f.d-real v.s.

$(\cdot,\cdot) : E^2 \longrightarrow \mathbb{R}$ positive def.

$\nu : E^* \xrightarrow{\sim} E$; $\alpha^\vee = \dfrac{2\nu(\alpha)}{(\alpha,\alpha)} \in E$

$(\cdot,\cdot)$ on E^*

$\forall \alpha \in E^* \setminus \{0\}$

$\alpha(\alpha^\vee) = 2$

$S_\alpha : E \to E$

$\phi \mapsto \phi - \alpha(\phi)\cdot\alpha^\vee$

$S_\alpha : E^* \to E^*$

$\gamma \longmapsto \gamma - \gamma(\alpha^\vee)\cdot\alpha$

$R \subset E^* \setminus \{0\}$ finite set s.t

(1) R spans E^*

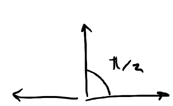
(2) $\alpha, c\alpha \in R \implies c = \pm 1$

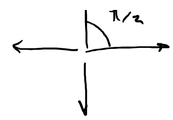
(3) $\alpha, \beta \in R \implies \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} = \beta(\alpha^\vee) \in \mathbb{Z}$

(4) $\forall \alpha \in R,\ S_\alpha(R) \subset R$

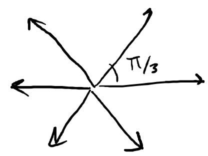
$\dim E = 2$, 4 possibilities

$A_1 \times A_1$

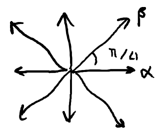




A_2

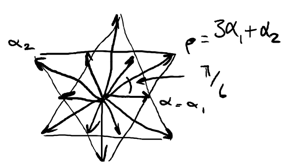


B_2



$|\beta|^2 = 2|\alpha|^2$

G_2



$|\beta|^2 = 3|\alpha|^2$

$\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2$, etc.

Positive / Negative roots

For $\alpha \in E^* \setminus \{0\}$, $H_\alpha = \mathrm{Ker}(\alpha) \subset E$
(hyperplane)

$E^\circ \setminus \bigcup_{\alpha \in R} H_\alpha$ — disconnected space

Note: for any connected component $\mathcal{C} \subset E^\circ$ (called a chamber)
and $\alpha \in R$, either $\alpha(x) > 0 \ \forall x \in \mathcal{C}$
or $\alpha(x) < 0 \ \forall x \in \mathcal{C}$

Choose a connected component $\mathcal{C}^\circ \subset E^\circ$,

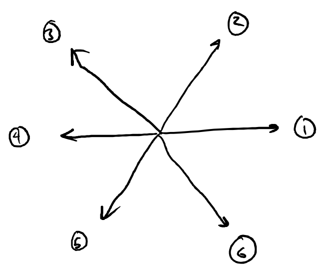
↑ fundamental chamber

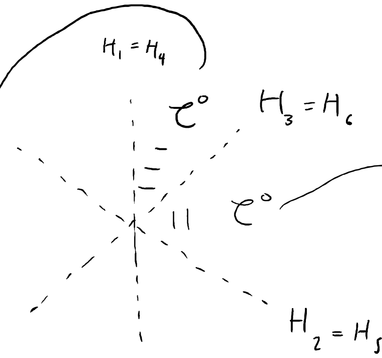
define $R_+ = \{\alpha \in R \mid \alpha(x) > 0 \ \forall x \in \mathcal{C}^\circ\}$

$R_- = \{\alpha \in R \mid \alpha(x) < 0 \ \forall x \in \mathcal{C}^\circ\}$

$R = R_+ \sqcup R_- \,; \quad R_- = -R_+ .$

eg A_2 :





6 connected components

$\{1, 2, 3\} = R_+$ $\qquad$ $\{1\ 2\ 6\} = R_+$

$\{4, 5, 6\} = R_-$ $\qquad$ $\{3\ 4\ 5\} = R_-$

Simple roots

$\alpha \in R$ is called a wall of a chamber $\mathcal{C} \subset E^\circ$

if $\alpha(x) > 0 \ \forall x \in \mathcal{C}$

and $\overline{\mathcal{C}} \cap H_\alpha$ is of co-dimension 1.

($\dim = \dim E - 1$).

Simple roots = walls of $\mathcal{C}^\circ$.

$$\begin{array}{c} \parallel \\ \{\alpha_i\}_{i \in I} \subset R_+ \end{array}$$

<u>Lemma</u> if $i \neq j \in I$; $c \in \mathbb{R}_{>0}$,

then $\alpha_i - c\alpha_j \notin R$.

<u>proof</u> α_i & α_j are not proportional.

If $\alpha = \alpha_i - c\alpha_j \in R$, then

$\alpha(x)$ either ≥ 0 on $\overline{\mathcal{C}^\circ}$

or ≤ 0 on $\overline{\mathcal{C}^\circ}$

but α on $\overline{\mathcal{C}^\circ} \cap H_{\alpha_i}$ is < 0

and α on $\overline{\mathcal{C}^\circ} \cap H_{\alpha_j}$ is > 0

$\square$

<u>Cor</u> $\forall\, i \neq j \in I$, let $a_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$.

Then $\alpha_{ij} \in \mathbb{Z}_{\leq 0}$

~~pf~~ $\alpha_i, \alpha_j \in R \Rightarrow S_{\alpha_i}(\alpha_j) = \alpha_j - \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \alpha_i$

$\cap$

R

$= \alpha_j - a_{ij} \alpha_i$

Lemma $\Rightarrow$ $a_{ij} \le 0$

$a_{ij} \in \mathbb{Z}$ by (3) $\square$

<u>Prop</u> $\{\alpha_i\}_{i \in I}$ is a basis of E^*.

pf (Linear independence). By contradiction

Assume $\sum_{t=1}^{p} c_t \alpha_{i_t} = 0$ is a linear rel$^{\underline{n}}$

where each $c_t \in \mathbb{R}_{>0}$ $(c_1 = 1)$

then $\underset{\uparrow \atop -\text{ on } \mathcal{C}^\circ}{-\alpha_{i_1}} = \underbrace{\sum_{t=2}^{p} c_t \alpha_{i_t}}_{\uparrow \atop +\text{ on } \mathcal{C}^\circ}$ contradiction

So if there is a liner rel$^{\underline{n}}$

among $\{\alpha_i\}_{i \in I}$, it must be of the form

$$\beta = \sum_{j=1}^{p} c_j \alpha_{i_j} = \sum_{k=p+1}^{q} d_k \alpha_{i_k}$$

where $c_j, d_k \in \mathbb{R}_{>0}$, $\{\alpha_{i_j}\}_1^p$ disjoint from $\{\alpha_{i_k}\}_{p+1}^q$.

$$(\beta, \beta) = \sum_{j,k} c_j d_k \underbrace{(\alpha_{i_j}, \alpha_{i_k})}_{\leq 0} \leq 0$$

but $(\cdot, \cdot)$ is positive definite.

So $\{\alpha_i\}_{i \in I}$ is lin. indep.

(spanning)

Since R spans E^*, it is enough to show that $\forall \alpha \in R$, $\{\alpha\} \cup \{\alpha_i\}_{i \in I}$ is linearly dependent.

If $\{\alpha\} \cup \{\alpha_i\}_{i \in I}$ is linearly independent

then we can find $x, y \in \mathcal{C}^\circ$ s.t. $\alpha(x) > 0$, $\alpha(y) < 0$.

contradiction.

(find x s.t.

$\alpha_i(x) = 1 \quad \forall i$

$\alpha(x) = 1$

| y s.t.

$\alpha_i(y) = 1$

$\alpha(y) = -1$) $\square$

Cartan matrix of R

$$A = (a_{ij})_{i,j\in I} \quad \text{where}$$

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \alpha_j(\alpha_i^\vee) \in \mathbb{Z}_{\leq 0} \quad \text{if } i \neq j$$

$$a_{ii} = 2$$

Rank 2 classification: (up to switching $i \leftrightarrow j$).

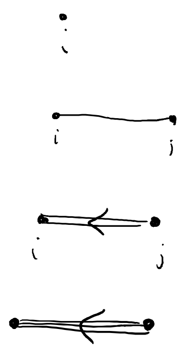
we have the following options for (a_{ij}, a_{ji})

$(0,0)$ no edge

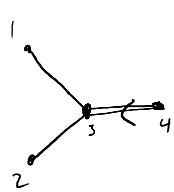
$(-1,-1)$ 'simple edge'

$(-2,-1)$ 'double edge + < sign' $|\alpha_j|^2 = 2|\alpha_i|^2$.

$(-3,-1)$ 'triple edge + < sign' $|\alpha_j|^2 = 3|\alpha_i|^2$.



The resulting graph is called the Dynkin diagram for R.



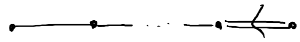
$$\approx \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

not positive-definite

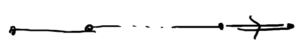
$$A\begin{bmatrix}1\\2\\1\end{bmatrix}=0$$

List of all connected Dynkin diagrams which arise from root systems:

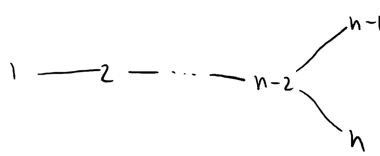
A_n



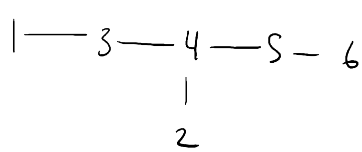
B_n



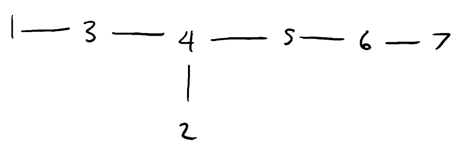
C_n



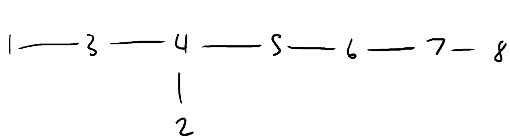
D_n



E_6



E_7



E_8

F_4

4

$1 \Rrightarrow 2$ G_2

$\text{Rank} = |I| = \dim E^*$

to prove classification,

get matrix of bad diagram,

find vector that kills it.