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Hyperplane Arrangements

V : n-dim $\mathbb{C}$-v.s.

$X \subset V^* \setminus \{0\}$ • $x \neq y \in X \Rightarrow x$ & y not proportional.

$\uparrow$ finite • X spans V^*.

$$\nabla = d - \sum_{x \in X} \frac{dx}{x} t_x$$

$\nabla F = 0$ is the following system of PDEs:

• if $\{x_1, \dots, x_n\}$ is a basis of V^*,

then $\forall x \in X$, $x = \sum_{i=1}^{n} n_i(x) x_i$

i.e. $dx = \sum_{i=1}^{n} n_i(x) dx_i$

Then $$\boxed{\frac{\partial F}{\partial x_i} = \sum_{x \in X} \frac{n_i(x)}{x} t_x F \qquad \forall i = 1, \dots, n}$$

Lemma (Kohno) This system is consistent iff

$\forall Y \subset X$ max'l s.t. Span(Y) is 2-dim'l, we have $\left[\sum_{y \in Y} t_y \, , \, t_z \right] = 0 \quad \forall z \in Y.$

<u>Example 1</u> $V \cong \mathbb{C}^n,$

$$V \supset X = \{ z_i - z_j \mid 1 \le i < j \le n \}$$

$$\left(\forall x \in X, \; H_x := \ker(x) \subset V \right)$$

$$H_{z_i - z_j} = \{ (u_1, \dots, u_n) \in \mathbb{C}^n \mid u_i = u_j \}$$

$$V_{reg} = V \setminus \bigcup_{i<j} H_{z_i - z_j}$$

= configuration space of n points in $\mathbb{C}$ (ordered)

- $Y \subset X$ max'l st $\dim(\operatorname{Span} Y) = 2$.

eg $\{ z_i - z_j \, , \, z_k - z_l \}$ i, j, k, l distinct

$$\{ z_i - z_j,\ z_j - z_k,\ z_i - z_k \}$$

Kohno's lemma $\rightsquigarrow$ rel$^{\underline{n}}$s for t_{ij}'s :

$$[t_{ij}, t_{k\ell}] = 0 ,$$

$$[t_{ij} + t_{jk} + t_{ik},\ t_{ij}] = 0$$

or

t_{jk}

or

t_{ik}

<u>Example 2</u> V is 2 dim'l, $x, y \in V^*$ is basis

$$X = \{x, y, x+y\}$$

$$\nabla = d - \underbrace{\left(\frac{dx}{x} t_1 + \frac{dy}{y} t_2 + \frac{d(x+y)}{x+y} t_3 \right)}_{A}$$

$dA = 0$ (because $\frac{dx}{x} = d(\log x)$)

$$A \wedge A = dx \wedge dy \left(\frac{t_1 t_2}{xy} + \frac{t_1 t_3}{x(x+y)} - \frac{t_2 t_1}{xy} - \frac{t_2 t_3}{y(x+y)} - \frac{t_3 t_1}{(x+y)x} + \frac{t_3 t_2}{(x+y)y} \right)$$

$$= dx \wedge dy \left(\frac{[t_1, t_2]}{xy} + \frac{[t_1, t_3]}{\quad} - \frac{[t_2, t_3]}{\quad} \right)$$

$$= dx \wedge dy \left(\frac{[t_1, t_2]}{xy} + \frac{[t_1, t_3]}{x(x+y)} - \frac{[t_2, t_3]}{y(x+y)} \right)$$

$$\mathcal{A} \wedge \mathcal{A} = 0 \iff (x+y)[t_1, t_2] + y[t_1, t_3] - x[t_2, t_3] = 0$$

$+t_2$

Coeff of x : $[t_1 + t_3, t_2] = 0$

Coeff of y : $[t_1, t_2 + t_3] = 0$

$+t_1$

Proof of Kohno's Lemma:

$$\mathcal{A} = \sum_{x \in X} \frac{dx}{x} t_x$$

To prove: $d\mathcal{A} - \mathcal{A} \wedge \mathcal{A} = 0$

$\iff$ (rank 2 commutation relations)

$d\mathcal{A} = 0$ in all cases (check!)

$$\mathcal{A} \wedge \mathcal{A} = 0 \Rightarrow x \, \mathcal{A} \wedge \mathcal{A} \Big|_{x=0} = 0$$

(this is $\iff$)

$$\mathcal{A} \wedge \mathcal{A} = \frac{1}{2} \sum_{y_1, y_2 \in X} \frac{dy_1 \wedge dy_2}{y_1 y_2} [t_{y_1}, t_{y_2}]$$

$$x \, \mathcal{A} \wedge \mathcal{A} \Big|_{x=0} = dx \wedge \sum_{y \in X} \frac{dy}{y}\Big|_{x=0} [t_x, t_y]$$

• $X \setminus \{x\}$ equivalence relⁿ

$y_1 \sim y_2$ if $y_1\big|_{x=0}$ is prop. to $y_2\big|_{x=0}$

i.e. if $\begin{cases} y_1 \in \mathrm{Span}\{x, y_2\} \\ y_2 \in \mathrm{Span}\{x, y_1\} \end{cases}$

$$\boxed{\left.\frac{dy_1}{y_1}\right|_{x=0} = \left.\frac{dy_2}{y_2}\right|_{x=0}}$$

• $X \setminus \{x\} = \underset{\substack{\cup \\ y_1}}{X_1} \sqcup \cdots \sqcup \underset{\substack{\cup \\ y_k}}{X_k}$ (equivalence classes)

y_k: $\boxed{\text{\& } x \text{, span } 2 \text{ dim}^{\text{l}} \text{ space}}$

$$x\,A \wedge A\Big|_{x=0} = \sum_{j=1}^{k} \left.\frac{dx \wedge dy_j}{y_j}\right|_{x=0} \Big[t_x, \sum_{y' \in X_j} t_{y'}\Big] = 0$$

$$\Leftrightarrow \Big[t_x, \sum_{y' \in X_j} t_{y'}\Big] = 0 .$$

□

Example t_{ij} = transposition $(i\ j) \in S_n$ $(1 \le i, j \le n,\ i \ne j)$

$$[t_{ij}, t_{kl}] = 0, \quad [t_{ij} + t_{jk} + t_{ik}, t_{ik}] = 0$$

$\rightsquigarrow$ a consistent system on $Y_n(\mathbb{C}) = \mathbb{C}^n \setminus$ diagonals

$$\nabla = d - \sum_{1 \le i < j \le n} \frac{d(z_i - z_j)}{z_i - z_j} (i\ j)$$

Root Systems (as examples of hyperplane arrangements).

(trigonometric analogue of Kohno's lemma? only for root systems)

Let E be a real (finite-dim) vector space together w/ a positive definite, symmetric, bilinear form:

$$(\cdot, \cdot) : E^2 \longrightarrow \mathbb{R}.$$

- use it to identify $E^* \xrightarrow[\nu]{\sim} E$

$$(\nu(\alpha), \phi) = \alpha(\phi) \qquad \forall\ \alpha \in E^*,\ \phi \in E.$$

- For every $\alpha \in E^*$, $\alpha \neq 0$, we have a linear map

$$S_\alpha : E \to E$$
$$\phi \mapsto \phi - \frac{2\alpha(\phi)}{(\alpha, \alpha)} \nu(\alpha) \qquad \text{(reflection in } \operatorname{Ker}\alpha)$$

$$S_\alpha : E^* \longrightarrow E^*$$
$$\gamma \longmapsto \gamma - \frac{2(\gamma, \alpha)}{(\alpha, \alpha)} \cdot \alpha$$

$S_\alpha(\gamma) = \gamma \quad \forall \gamma$ s.t. $(\gamma, \alpha) = 0$

$S_\alpha(\alpha) = -\alpha$

$S_\alpha^2 = id$

<u>Defn</u> A root system is a finite set of nonzero elements of E^*: $R \subset E^* \setminus \{0\}$ (finite) s.t

(1) $\alpha, \beta \in R$; $\alpha = c\beta \Rightarrow c = \pm 1$

(2) R spans E^*

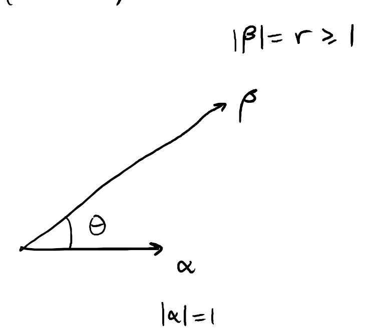
(3) (integrality): $\alpha, \beta \in R \Rightarrow \dfrac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$

(4) $\forall \alpha \in R$, $S_\alpha(R) \subset R$

<u>Examples in 2D</u>

Pick $\alpha, \beta \in R$ s.t. $\angle_\alpha^\beta$ is acute (use (4))

$|\beta| \geq |\alpha|$



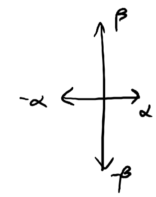
<u>Integrality</u>: $2 r \cos\theta \in \mathbb{Z}$
$2 \ldots$ $\Big\}$ $4\cos^2\theta \in \mathbb{Z}$

Integrality: $\left.\begin{array}{l} 2r\cos\theta \in \mathbb{Z} \\ \frac{2}{r}\cos\theta \in \mathbb{Z} \end{array}\right\} 4\cos^2\theta \in \mathbb{Z}_{\geq 0}$

$4\cos^2\theta = 0, 1, 2, 3,$ or $\cancel{4}$

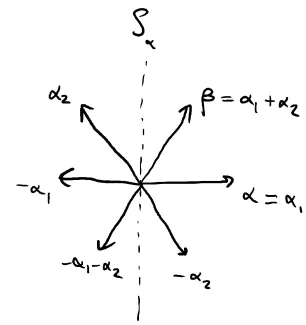
α & β not proportional

$4\cos^2\theta = 0$: $\theta = \frac{\pi}{2}$ $\Rightarrow$ root system =



reducible.

$4\cos^2\theta = 1$: $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$ $\Rightarrow$ $r=1$:



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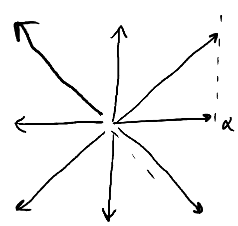
$4\cos^2\theta = 2$: $\cos\theta = \frac{1}{\sqrt{2}}$, $\theta = \frac{\pi}{4}$ $\Rightarrow$ $r = \sqrt{2}$



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last case: $r = \sqrt{3}$, $\theta = \frac{\pi}{6}$, ...