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ODEs over $\mathbb{C}$.

$$\boxed{\frac{dF}{dz} = A(z)\,F(z)}$$

Let $D \subset \mathbb{C}$ be an open connected set.

Let $n \geq 1$.

given $A : D \longrightarrow M_{n\times n}(\mathbb{C})$ holomorphic (meromorphic)

F unknown $D \xrightarrow{F} GL_n(\mathbb{C})$

$(F : D \longrightarrow M_{n\times n}(\mathbb{C})$ and $\det F \not\equiv 0)$

Facts:

(1) $\frac{d}{dz}(F\cdot G) = F'\cdot G + F\cdot G'$

(2) $\frac{d}{dz}\left(F(z)^{-1}\right) = -F(z)^{-1}\left(\frac{d}{dz}F(z)\right)F(z)^{-1}$

(since $F\cdot F^{-1} = \mathrm{id}$

$F'\cdot F^{-1} + F\cdot(F^{-1})' = 0$)

(3) if F_1, F_2 are two (invertible) solutions of

$F' = AF$ then

$F_1(z)^{-1} F_2(z)$ is ind. of z:

$$\left(\frac{d}{dz}\left(F_1^{-1} F_2\right) = -F_1^{-1}\left(A \cancel{F_1}\right)\cancel{F_1^{-1}} F_2 + F_1^{-1}\left(A F_2\right) = 0 \right)$$

Assume D = disc around 0. $A : D \setminus \{0\} \longrightarrow M_{n\times n}(\mathbb{C})$.

We say $0 \in D$ is an ordinary point if A is holomorphic at 0.

So $A(z) = A_0 + A_1 z + A_2 z^2 + \cdots$ $\qquad (*)\ \frac{d}{dz} F = A \cdot F$

where $A_i \in M_{n\times n}(\mathbb{C})$ $\forall i$.

Thm: in this case, $\exists!$ $F(z)$ solution of $(*)$ s.t. $F(0) = \mathrm{Id}_{n\times n}\ (= 1)$

proof: $F(z) = F_0 + F_1 z + F_2 z^2 + \cdots$

$(F_0 = 1)$.

(Frobenius)

$(*) \longleftarrow$ coeff of z^n

$$\boxed{(n+1)\, F_{n+1} = \sum_{k=0}^{n} A_k F_{n-k}}$$

we have existence & uniqueness of formal soln,

now show it converges:

<u>exercise:</u> if $\sum_{k=0}^{\infty} a_k z^k$ is a power series ($a_k \in \mathbb{C}$), r is its radius of convergence ($r \neq 0$).

Then $\{f_n\}_{n \geq 0}$ defined by $\begin{cases} f_0 = 1 \\ f_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k f_{n-1-k} \end{cases}$

gives a power series $\sum_{k=0}^{\infty} f_k z^k$ with r.o.c. r as well.

Passing to norms gives convergence of matrix series. □

We say $0 \in D$ is a regular singular point if $A(z)$ has a simple (order 1) pole at 0.

(Fuchsian / Logarithmic singularity).

$$A(z) = \frac{\Lambda}{z} + A_0 + A_1 z + A_2 z^2 + \dots$$

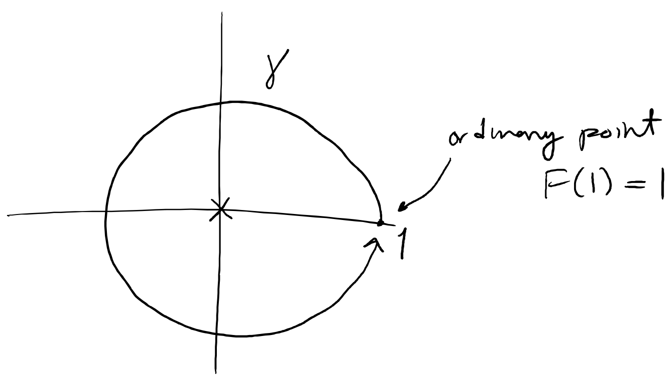
Example: $F'(z) = \frac{\Lambda}{z} F(z)$ $\qquad \Lambda \in M_{n \times n}(\mathbb{C})$.

solved by $F(z) = z^{\Lambda}$

$= \exp(\Lambda \ln(z))$

this is not single-valued.

$$\ln : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \longrightarrow \mathbb{C}, \quad \ln(1) = 0.$$



$\gamma : [0,1] \longrightarrow \mathbb{C}$

$t \longmapsto e^{2\pi i t}$

$\tilde{F}(z)$ new solution near 1, continued around the loop

$$\tilde{F}(z) = z^{\Lambda} \cdot \exp(2\pi i \Lambda)$$

$$\mu(\gamma) = F^{-1}\tilde{F} = \exp(2\pi i \Lambda).$$

$\mu \downarrow$ w.r.t. F $\qquad \exp(2\pi i\Lambda) \in GL_n(\mathbb{C})$.

monodromy

$\gamma : [0,1] \longrightarrow X$ $\qquad$ Diff'l eqns on X:

$\gamma(0) = \gamma(1) = x_0,$ $\qquad \nabla F = 0.$

(over n-dim'l v.s.)

- Solve $\nabla F = 0$ near x_0.
- $\tilde{F}$ = analytic continuation of F along γ.
- $\boxed{\mu_{F;x_0}(\gamma) \overset{\text{defn}}{=} F^{-1}\tilde{F}}$
- $\mu_{F;x_0} : \pi(X, x_0) \longrightarrow GL_n(\mathbb{C})$

  is a group hom.

$$F'(z) = A(z)\,F(z)$$

$$A(z) = \frac{\Lambda}{z} + A_0 + A_1 z + A_2 z^2 + \cdots$$

look for sol$^{\underline{n}}$ of the form

$$F(z) = \underbrace{H(z)}_{\text{unknown.}} \, z^{\Lambda} \, .$$

We get

$$H'(z) \cdot \cancel{z^{\Lambda}} + H(z) \cdot \frac{\Lambda}{z} \cancel{z^{\Lambda}}$$

$$= \left(\frac{\Lambda}{z} + A_{reg}(z) \right) H(z) \cdot \cancel{z^{\Lambda}}$$

so

$$H'(z) = \frac{[\Lambda, H(z)]}{z} + A_{reg}(z)\, H(z) \qquad (\dagger)$$

where $[X, Y] = XY - YX$.

this eqn has 0 as an ordinary pt provided $H_0 = 1$

If $H(z) = H_0 + H_1 z + \cdots$

Coeff of z^{-1} in $(\dagger)$. $\quad 0 = 0 \quad \checkmark$

Coeff of z^m : $\quad (m+1)\, H_{m+1} = [\Lambda, H_{m+1}] + \sum_{k=0}^{m} A_k H_{m-k}$
$(m \geq 0)$

$$(m+1 - \mathrm{ad}(\Lambda)) \cdot H \quad = \sum^{m} A \; H$$

$$(m+1 - \mathrm{ad}(\Lambda)) \cdot H_{m+1} = \sum_{k=0}^{m} A_k H_{m-k}$$

$$\mathrm{ad}(\Lambda) : M_{n\times n}(\mathbb{C}) \longrightarrow M_{n\times n}(\mathbb{C})$$

$$Y \longmapsto [\Lambda, Y] = \Lambda Y - Y\Lambda$$

$$\left(\text{eg } \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} ; \quad Y \xmapsto{\mathrm{ad}(\Lambda)} \left((\lambda_i - \lambda_j) y_{ij} \right) \right).$$

<u>Defn</u>: Λ is called non-resonant if its eigenvalues do not differ by nonzero integers.

<u>Examples</u> $\Lambda = c \cdot \mathrm{Id}$ ✓

Λ nilpotent ✓

$$\Lambda \in M_{n\times n}(\mathbb{C})$$

$$\Downarrow$$

$\hbar \Lambda$ is non-resonant for generic $\hbar$.

<u>Thm</u>. Assuming Λ is non-resonant,

(Frobenius)

$$F'(z) = A(z) F(z) \quad \text{has a unique sol'n}$$

1 (Frobenius) $F'(z) = A(z)F(z)$ has a unique sol'n of the form $H(z)\cdot z^{\Lambda}$ where H is holomorphic near 0; $H(0)=1$.

[Fundamental Solution]

$$A(z) = \frac{\Lambda}{z} + A_0 + A_1 z + A_2 z^2 + \dots$$

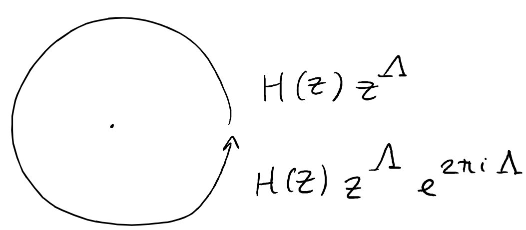
Ordinary Case: $$F_m = \frac{1}{m}\sum_{j=0}^{m-1} A_j F_{m-1-j}$$

Now: (reg. sing. case) $$H_m = (m - \mathrm{ad}(\Lambda))^{-1} \sum_{j=0}^{m-1} A_j H_{m-1-j}$$

$$\exists K \in \mathbb{R}_{>0} \text{ s.t. } \left\| (m-\mathrm{ad}(\Lambda))^{-1} \right\| < \frac{K}{m}.$$

Cor: w.r.t $F(z) = H(z)\cdot z^{\Lambda}$ sol.^{n}

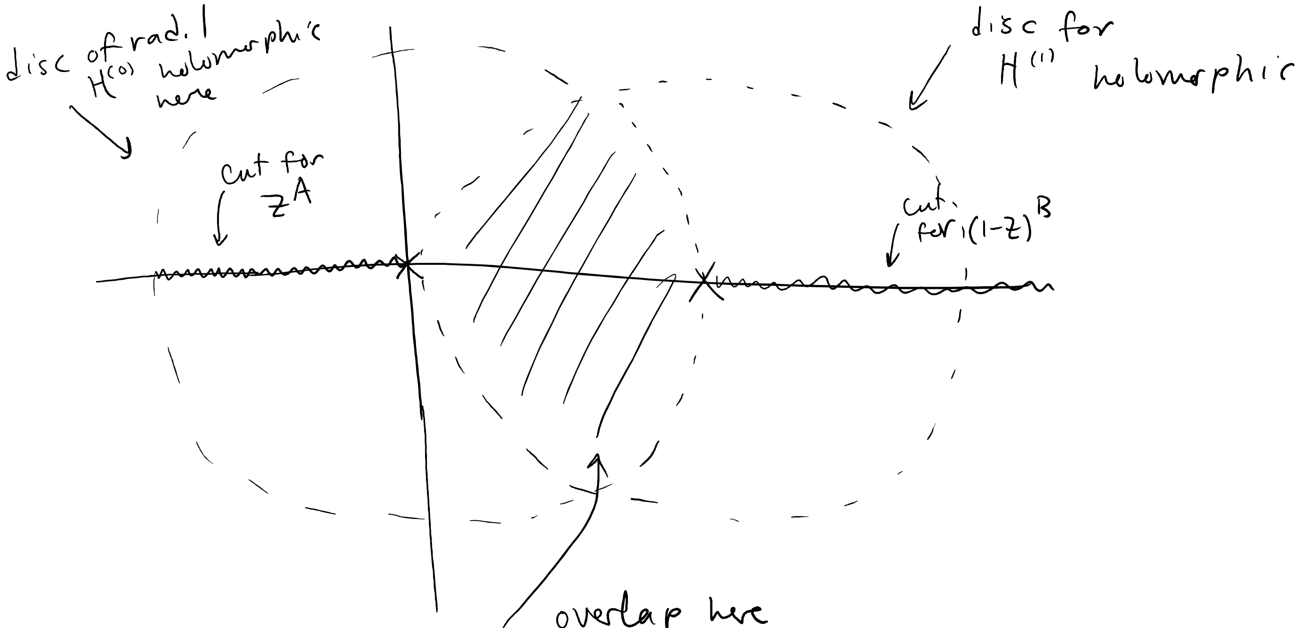
$$\mu(\circlearrowright) = \exp(2\pi i \Lambda) \quad \text{still}$$



Consider $$\frac{dF}{dz} = \left(\frac{A}{z} + \frac{B}{z-1} \right) F$$

$A, B \in M_{n\times n}(\mathbb{C})$, non-resonant.

We have a soln
$$\boxed{F^{(0)}(z) = H^{(0)}(z) \cdot z^A} \qquad \boxed{F^{(1)}(z) = H^{(1)}(z) \cdot (1-z)^B}$$



$$F^{(0)}(z) = F^{(1)}(z) \cdot K$$

for $K \in M_{n\times n}(\mathbb{C})$ called

"Drinfield associator"

$$\mu_{F^{(0)}}\left(\cdot_0 \right) = e^{2\pi i A}$$

$$\mu_{F^{(1)}}\left(\cdot_1 \right) = e^{2\pi i B}$$

but $$\mu_{F^{(0)}}\left(\cdot_1 \right) = \mu_{F^{(1)} \cdot K}\left(\cdot_1 \right)$$

$$= K^{-1} \mu_{F^{(1)}}\left(\cdot_1 \right) K$$