8/21

Wednesday, August 21, 2019 13:49

No class next week.

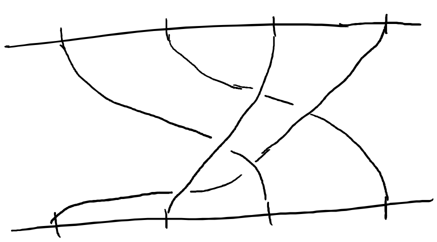
Kohno-Drinfield Theorem:

(connection btwn diff-l eqns & quantum groups).

Asserts equivalence of two rep^{n}s of Artin's braid group B_n $(n \geq 2)$

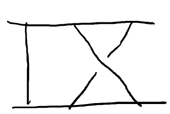
1. Artin's braid group.

Let $n \in \mathbb{Z}_{\geq 2}$. then B_n is defined as the set of all braids on n strands

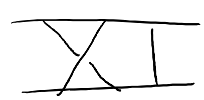


group operation: concatenation.

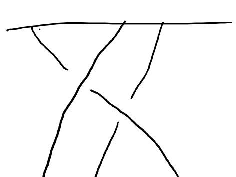
b_1 :



b_2 :



$b_1 b_2$:



Inverse: switch under $\longleftrightarrow$ over

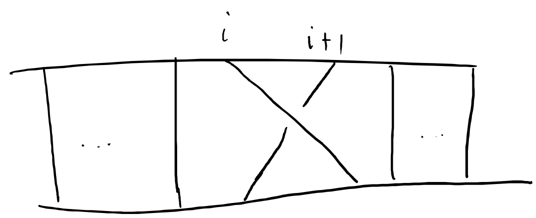
bottom $\longleftrightarrow$ top

Artin's theorem: B_n admits a presentation:

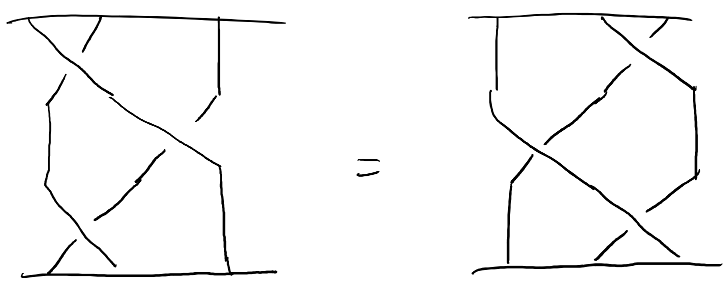
$$\left\langle T_i \quad 1 \le i \le n-1 \;\middle|\; \begin{array}{l} T_i T_j = T_j T_i \quad \text{if } |i-j|>1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \forall\, 1 \le i \le n-2 \end{array} \right\rangle$$

Recall: adding rel^n T_i^2 gives S_n.

Here, T_i =



2^{nd} rel^n:



This theorem means: in order to construct

a gp hom $\varphi : B_n \longrightarrow G$,

we just need to find $n-1$ elements in G
which satisfy those relns.
(the "braid relations")

A representation of B_n on a vector space V (over $\mathbb{C}$)
is a group hom $B_n \longrightarrow GL(V)$.

K-D theorem.

Monodromy of ∇_{KZ} = R-matrix of quantum group.

2. B_n is fundamental group of $\mathrm{Conf}_n(\mathbb{C})$

configuration space
of n (unordered, distinct)
points in $\mathbb{C}$.

$$Y_n(\mathbb{C}) \subset \mathbb{C}^n$$
$$\|$$
$$\mathbb{C}^n \setminus \{\underline{z} : z_i = z_j \text{ for some } i \neq j\}$$

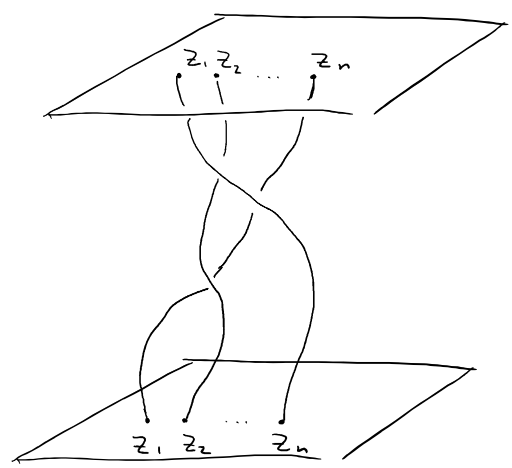
S_n acts on $Y_n(\mathbb{C})$ $\quad (S_n \circlearrowright Y_n(\mathbb{C}))$

$$\mathrm{Conf}_n(\mathbb{C}) = Y_n(\mathbb{C}) / S_n .$$

$$B_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}), \underset{\text{some base point}}{P_0})$$

A point in $\mathrm{Conf}_n(\mathbb{C})$

$\gamma : [0,1] \longrightarrow \mathrm{Conf}_n(\mathbb{C})$



representation of $\pi_1(X, x_0)$ can be constructed using <u>monodromy</u>.

KZ equations.

Let V be a finite-dimensional vector space over $\mathbb{C}$.

$\Omega \in \mathrm{End}(V \otimes V)$.

Notation $\Omega_{ij} \in \mathrm{End}(\overbrace{V \otimes \cdots \otimes V}^{n \text{ times}})$

$\forall\, i, j \in \{1, \dots, n\}$, $i \neq j$.

For a function $F : Y_n(\mathbb{C}) \longrightarrow \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$

$$\boxed{KZ_n \text{ equations: } \quad \frac{\partial F}{\partial z_i} = \sum_{\substack{j \neq i \\ 1 \leq j \leq n}} \frac{\Omega_{ij}}{z_i - z_j} F}$$

Thm (KZ) if $\Omega_{21} = \Omega$ and

$\Omega_{12} + \Omega_{23} + \Omega_{13}$ commutes with

Ω_{12}, Ω_{23}, and Ω_{13}

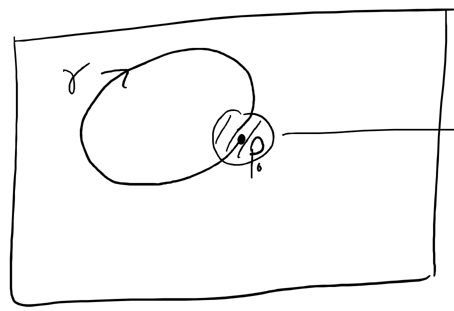
Then KZ_n is consistent & S_n-equivariant

$$\left.\begin{aligned} &P \in \mathrm{End}(V \otimes V) \\ &V \otimes V \longrightarrow V \otimes V \\ &v \otimes w \longmapsto w \otimes v \end{aligned}\right\} \quad \begin{aligned} &1^{st} \text{ statement is} \\ &\Omega \text{ commutes w/ } P \end{aligned}$$

$$\left(\Omega_{21} = P\,\Omega_{12}\,P\right).$$

What is monodromy?

X $\qquad\qquad$ $\nabla F = 0$



let ψ be a solution of $\nabla \psi = 0$ near p_0

$$\tilde{\psi} = \text{analytic continuation of } \psi \text{ along } \gamma$$

$$\mu(\gamma) = \tilde{\psi}^{-1}\psi \qquad \text{constant (ind. of } z).$$

$$\mu : \gamma \longmapsto C_\gamma \in GL(W)$$

<u>Quantum Groups:</u>

- associative algebra $\mathcal{A}$
- $R \in \mathcal{A} \otimes \mathcal{A}$ invertible

s.t. $\boxed{R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}}$

assume we have all of this

$$\boxed{R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}}$$

Yang-Baxter equation

in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

Then $\forall$ rep^n $\underbrace{V \text{ of } \mathcal{A}}$ and $n \geq 2$ we have a group hom

an alg. hom $\rho : \mathcal{A} \longrightarrow \operatorname{End}(V)$

$$B_n \xrightarrow{\sigma} GL(V^{\otimes n})$$

$$T_i \longmapsto (i \ \ i{+}1)\left(1^{\otimes i-1} \otimes (\rho \otimes \rho(R)) \otimes 1^{\otimes n-i-1}\right)$$

elt of S_n

<u>Proof</u>: By Artin's theorem, we need to check

$$\sigma(T_i)\, \sigma(T_j) = \sigma(T_j)\, \sigma(T_i) \qquad \text{if } |i-j| > 1$$

this is clear.

$$\sigma(T_1 T_2 T_1) = (12) R_{12} \ (23) R_{23} \ (12) R_{12}$$

$$= (12)(23)(12) R_{23} R_{13} R_{12}$$

$$= (13) R_{23} R_{13} R_{12}$$

$$\sigma(T_2 T_1 T_2) = (13)\, R_{23} R_{13} R_{12} \quad \text{as well}$$

K-D Theorem:

$$\pi_1(\mathrm{Conf}_n \mathbb{C}, p_0) = B_n = \langle T_i \ \ 1 \le i \le n-1 \mid \cdots \rangle$$

repns of B_n via monodromy of KZ eqns

Quantum Groups (R-matrices)

this part

(each side gives a braided tensor structure)