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Dynamical Considerations (Etingoff + Varchenko)

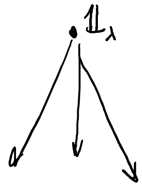
* $q = e^{\hbar/2} \in \mathbb{C}^\times$ not a root of unity.

$M_\lambda \circlearrowleft U_q(\mathfrak{g})$

generated as module by $\mathbb{1}_\lambda \in M_\lambda$

$h\mathbb{1}_\lambda = \lambda(h)\mathbb{1}_\lambda$; $E_i \mathbb{1}_\lambda = 0 \quad \forall i \in I$.

$\mathbb{1}_\lambda$ highest wt vector (in $M_\lambda[\lambda]$)



$\leftarrow M_\lambda[\lambda - \gamma]$, $\gamma \in Q_+ \setminus \{0\}$.

Prop Let V be a f.d. repn of $U_q(\mathfrak{g})$, $\mu \in P(V)$.

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(M_\lambda, M_{\lambda-\mu} \otimes V) \cong V[\mu].$$

$$\varphi \longmapsto \overbrace{\text{coeff of } \mathbb{1}_{\lambda-\mu} \text{ in } \varphi(\mathbb{1}_\lambda)}^{\in V[\mu]}, \quad \varphi(\mathbb{1}_\lambda) \in (M_{\lambda-\mu} \otimes V)[\lambda]$$

$$\varphi_\lambda^v \longleftarrow v$$

Fusion Operator

$U_q(\mathfrak{g}) \circlearrowleft V_1, V_2$ f.d.

$v_1 \in V_1[\mu_1]$, $\quad v_2 \in V_2[\mu_2]$.

$$M_\lambda \xrightarrow{\varphi^{v_2}_\lambda} M_{\lambda-\mu_2} \otimes V_2 \xrightarrow{\varphi^{v_1}_{\lambda-\mu_2} \otimes \mathrm{id}} M_{\lambda-\mu_1-\mu_2} \otimes V_1 \otimes V_2$$

$$J_{V_1,V_2}(\lambda) \cdot v_1 \otimes v_2 \overset{\text{def}}{=} \text{coeff of } \mathbb{1}_{\lambda-\mu_1-\mu_2} \text{ in}$$

$$\left((\varphi^{v_1}_{\lambda-\mu_2} \otimes \mathrm{id}) \circ \varphi^{v_2}_\lambda \right)(\mathbb{1}_\lambda)$$

$$= \left\langle \mathbb{1}_{\lambda-\mu_1-\mu_2} \,\middle|\, (\varphi^{v_1}_{\lambda-\mu_2} \otimes \mathrm{id}) \circ \varphi^{v_2}_\lambda \,\middle|\, \mathbb{1}_\lambda \right\rangle$$

$$J_{V_1,V_2}(\lambda) : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$$ <u>fusion operator</u>

<u>Some properties</u>

$J_{V_1,V_2}(\lambda)$ is lower triangular w/ 1's on diagonal, rational ($\mathrm{End}(V_1 \otimes V_2)$-valued) function of q^λ.

<u>Thm</u> $\lim_{q^\lambda \to \infty} J(\lambda) = (\overline{R})^{-1}$.

Twist / Cocycle eqn

Let V_1, V_2, V_3 be f.d. reps. Then

$$J_{V_1 \otimes V_2, V_3}(\lambda)\, J_{V_1, V_2}(\lambda - wt_3) = J_{V_1, V_2 \otimes V_3}(\lambda)\, J_{V_2, V_3}(\lambda)$$

$(wt_3(v_3) = \mu_3$ for $v_3 \in V_3[\mu_3])$.

Dynamical Weyl Group

Singular vectors: $\mathbb{1}_\lambda \in M_\lambda$. If $\lambda \in \mathbb{Z}_{\geq 0}$, then $F^{(\lambda+1)} \mathbb{1}_\lambda$ is singular

In higher rank: if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$, $\underbrace{F_i^{(\lambda(h_i)+1)} \mathbb{1}_\lambda}$ is singular.

$$\cap$$

$$M_\lambda[\underbrace{\lambda - (\lambda(h_i)+1)\alpha_i}_{s_i \cdot \lambda}]$$

(shifted action).

Let $\rho \in \mathfrak{h}^*$ s.t. $\rho(h_i) = 1 \quad \forall i$.

Then $w \cdot \lambda \overset{\text{def}}{=} w(\lambda + \rho) - \rho$ (shifted action).

Let $w \in W$; $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression.

Set $\beta_1 = \alpha_{i_1}$, $\beta_2 = s_{i_1}(\alpha_{i_2})$, $\ldots$, $\beta_\ell = s_{i_1} \cdots s_{i_{\ell-1}}(\alpha_{i_\ell})$

$$n_j = \frac{2}{(\beta_j, \beta_j)} (\lambda + \rho, \beta_j) \in \mathbb{Z}_{\geq 0} \quad \text{(assume)}$$

if not, no singular vector.

Then $F_{i_\ell}^{(n_\ell)} \cdots F^{(n_1)} \mathbb{1}_\lambda \in M_\lambda[w \cdot \lambda]$ is again a singular vector.

Also, it's indep of reduced expression

↳ called Verma identities

Dynamical Weyl gp

- sl_2 $\quad U_q(sl_2) \underset{fd}{\circlearrowright} V$; $\quad v \in V[\mu]$ $\quad \lambda \in \mathbb{Z}_{\geq 0}$, $|\lambda| >> 0$

$$\begin{array}{ccc} M_\lambda & \longrightarrow & M_{\lambda-\mu} \otimes V \\ \cup & & \cup \\ M_{-\lambda-2} & \longrightarrow & M_{-\lambda+\mu-2} \otimes V \end{array}$$

$$\underset{\text{m}}{A_{s;V}}(\lambda) \cdot v = \left\langle F^{(\lambda-\mu+1)} \mathbb{1}_{\lambda-\mu} \middle| \varphi_\lambda^v \middle| F^{(\lambda+1)} \mathbb{1}_\lambda \right\rangle$$

$V[-\mu]$

In general, $w = s_{i_1} \cdots s_{i_\ell}$

$$\rightsquigarrow A_{w;V}(\lambda) = A_{s_{i_1};V}(s_{i_1} \cdots s_{i_\ell} \cdot \lambda) \cdots A_{s_{i_{\ell-1}};V}(s_{i_\ell} \cdot \lambda) \cdot A_{s_{i_\ell};V}(\lambda)$$

$$A_{w;V}(\lambda) \cdot v = \left\langle \mathbb{1}_{w \cdot (\lambda-\mu)} \,\middle|\, \varphi^v_\lambda \,\middle|\, \mathbb{1}_\lambda \right\rangle$$ indep of reduced exp.

Coproduct Identity

$$A_{w;V_1 \otimes V_2}(\lambda) \cdot J_{V_1,V_2}(\lambda) = J_{V_1,V_2}(w \cdot \lambda)\left(A_{w;V_1}(\lambda - wt_2) \otimes A_{w;V_2}(\lambda)\right)$$

<u>for sl_2</u>,

Lusztig elt.

$$A_{s_i L_m}(\lambda) \xrightarrow[q^\lambda \to \infty]{} (-1)^m S$$

<u>Cor</u> (1) $\{S_i\}_{i \in I}$ satisfy Braid rel^{n}s.

(2) $\Delta(S_i) = (S_i \otimes S_i)\,\overline{R_i}$