11/6

Wednesday, November 6, 2019 13:52

Summary of quantum groups

- $U_\hbar(\mathfrak{g})$ generators & rel^{n}s
- Hopf alg: Δ, S, ε.
- R-matrix: canonical tensor of Drinfeld pairing

$\longrightarrow$ $(U_\hbar(\mathfrak{g}), R)$ = quasi-triangular Hopf alg.

- Braid group action $B_W \circlearrowright U_\hbar(\mathfrak{g})$
  algebra automorphisms

$$\forall i \in I, \quad T_i : U_\hbar(\mathfrak{g}) \longrightarrow U_\hbar(\mathfrak{g})$$

$$E_i \longmapsto -F_i K_i$$

$$F_i \longmapsto -K_i^{-1} E_i$$

$$h \longmapsto s_i(h)$$

(something complicated on E_j, F_j if $j \neq i$)

<u>R_K</u>: "$T_i x = S_i \, x \, S_i^{-1}$"

where $S_i = \exp_{q_i}(q_i^{-1} E_i K_i^{-1}) \exp_{q_i}(-F_i) \exp_{q_i}(q_i E_i K_i) \, q_i^{\frac{h_i(h_i+1)}{2}}$

(Lusztig)

$\{T_i\}$ and $\{S_i\}$ satisfy braid relⁿs.

Poincaré - Birkhoff - Witt Theorem (PBW)

$\mathfrak{a}$: Lie alg. $\{x_j\}_{j\in J}$ a basis of $\mathfrak{a}$ and a total order $<$ on J

Then $\{x_{j_1}^{n_1} \cdots x_{j_r}^{n_r} : n_1, \dots, n_r \in \mathbb{Z}_{\geq 0},\ j_1 < j_2 < \cdots < j_r \in J\}$

forms a basis for $\mathcal{U}(\mathfrak{a})$.

$\rightsquigarrow U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]$ as $\mathbb{C}[[\hbar]]$-modules.

A formula for $\mathcal{R}$ (Kirillov - Reshetikhin)

$$\boxed{\mathcal{R} = q^{\Omega_0} \overrightarrow{\prod_{\alpha \in R_+}} \exp_{q_\alpha}\left((q_\alpha - q^{-1}) F_\alpha \otimes E_\alpha\right)}$$

- where $\Omega_0 \in \mathfrak{h} \otimes \mathfrak{h}$ is canonical tensor of $(h_i, h_j) = \frac{a_{ij}}{d_j}$.
- Normal ordering on R_+:

Let $w_0 \in W$ be the longest element.

Let $w_0 = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression. $\ell = \ell(w_0) = |R_+|$

$\beta_1 = \alpha_{i_1}$, $\beta_2 = s_{i_1}(\alpha_{i_2}), \ldots, \beta_\ell = s_{i_1} \cdots s_{i_{\ell-1}}(\alpha_{i_\ell}) \in R_+$,

so $R_+ = \{\underbrace{\beta_1, \ldots, \beta_\ell}_{\longrightarrow \text{ total order}}\}$,

- $E_{\beta_1} = E_{i_1}$, $E_{\beta_2} = T_{i_1}(E_{i_1}), \ldots, E_{\beta_\ell} = T_{i_1} \cdots T_{i_{\ell-1}}(E_{i_\ell})$

similarly for F_α's.

<u>Coproduct identity</u> (for sl_2)

$$\Delta(S) = (S \otimes S) \cdot \underbrace{\exp_q\left((q - q^{-1}) F \otimes E\right)}_{\bar{R} = q^{-\frac{H \otimes H}{2}} R}$$

Representations of $U_\hbar(\mathfrak{g})$

V: f.d. v.s./$\mathbb{C}$. Consider $U_\hbar(\mathfrak{g}) \subset V[[\hbar]]$ "finite-dim'l"

$V[[\hbar]] = V \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$

representation theories of $U_\hbar(\mathfrak{g})$ and $U(\mathfrak{g})$ are exactly the same.

Irred. f.d. repn / iso $\longleftrightarrow$ $P_+ = \{\gamma \in \mathfrak{h}^* \mid \gamma(h_i) \in \mathbb{Z}_{\geq 0} \ \forall i \in I\}$

Ψ $\qquad$ Ψ

$$\begin{array}{ccc} \diagup \text{iso} & & \ \ \lbrace \gamma \in \mathfrak{h} \mid \gamma(h_i) \in \mathbb{Z}_{\geq 0} \ \forall i \in I \rbrace \\ \Psi & & \Psi \\ L_\lambda & \longleftrightarrow & \lambda \end{array}$$

L_λ has a non-zero cyclic vector v s.t.

$$\left| \begin{array}{ll} h \cdot v = \lambda(h) v & \forall h \in \mathfrak{h} \\ E_i v = 0 & \forall i \in I \\ F_i^{\lambda(h_i)+1} v = 0 & \forall i \in I \end{array} \right.$$

since $\underset{\text{as algebras}}{U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]} \circlearrowright V[[\hbar]]$.

Dynamical Versions - Due to Varchenko & Etingof

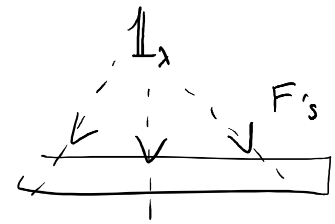
Verma Modules For any $\lambda \in \mathfrak{h}^*$, define M_λ to be the (infinite-dim'l) repn

$$U_\hbar(\mathfrak{g}) \Big/ \begin{array}{l} \text{left ideal gen by} \\ \left\lbrace \begin{array}{ll} h - \lambda(h) & \forall h \in \mathfrak{h} \\ E_i & \forall i \in I \end{array} \right\rbrace \end{array} = M_\lambda$$

$M_\lambda \ni$ (unique up to scalar) vector $\mathbb{1}_\lambda$

which generates M_λ as a repn,

$$\begin{cases} h \cdot \mathbb{1}_\lambda = \lambda(h) \mathbb{1}_\lambda & \forall h \in \mathfrak{h} \\ E_i \cdot \mathbb{1}_\lambda = 0 & \forall i \in I \end{cases}$$



Universal property:

singular vectors of wt λ

$$\mathrm{Hom}_{U_\hbar(\mathfrak{g})}(M_\lambda, V) \cong V[\lambda]^{\mathfrak{n}^+}$$

$$\left(\text{recall } V[\lambda]^{(\mathfrak{n}^+)} = \left\{ v \in V \;\middle|\; \begin{array}{l} h \cdot v = \lambda(h) \cdot v \\ (E_i \cdot v = 0) \end{array} \right\}\right.$$

- $M_\lambda = \bigoplus_{\mu \in Q_+} M_\lambda[\lambda - \mu]$ $\qquad \left(Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \right)$

$\dim M_\lambda[\lambda - \mu]$ = Kostant's partition fn

= # of ways of writing μ as sum of positive roots

$< \infty$