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Quantum Group $U_h(\mathfrak{g})$

- Defined Hopf Alg $\tilde{U}$ assoc. to Cartan matrix
- $\tilde{U}/_{rad} =: U_h(\mathfrak{g})$.

  rad: radical of $(\cdot,\cdot) : U^{\leq 0} \times U^{\geq 0} \longrightarrow U$

- $(\cdot,\cdot)$ descends to a non-degen pairing.

  $\rightsquigarrow R \in U^{\leq 0} \otimes U^{\geq 0}$ satisfies cabling & intertwining eq^{n}s.

  $\rightsquigarrow U_h(\mathfrak{g})$ is a quasi-Δ Hopf alg.

Serre Relations: $i \neq j \in I$. Let $m = 1 - a_{ij}$.

$$\Theta_{ij}^{+} = \sum_{s=0}^{m} (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_q E_i^{m-s} E_j E_i^{s} \in rad^{\geq 0}$$

(similarly Θ_{ij}^{-} for F's)

$$\left([l]_p = \frac{p^l - p^{-l}}{p - p^{-1}} \right)$$

Lemma Let V be a f.d v.s $/\mathbb{C}$.

Let $\rho : \tilde{U}_\hbar \longrightarrow \mathrm{End}(V)[[\hbar]]$ be an alg hom

Then $\rho(\theta_{ij}^{\pm}) = 0 \quad \forall\, i \neq j \in I.$

So Serre reln's usually hold automatically.

<u>Weyl gp action</u> : sl_2 -case

$$s = \exp(e)\exp(-f)\exp(e) \circlearrowright V$$

where e's & f's act locally nilpotently.

$$s : V[\mu] \longrightarrow V[-\mu]$$
$$v \in V[\mu]$$

$$s \cdot v = \sum_{\substack{a,b,c \geq 0 \\ b-a-c=\mu}} (-1)^b \frac{e^a}{a!} \frac{f^b}{b!} \frac{e^c}{c!} \cdot v$$

Lusztig Elt:

$$S = \exp_{q^{-1}}(q^{-1}EK^{-1}) \exp_{q^{-1}}(-F) \exp_{q^{-1}}(qEK)\, q^{\frac{H(H+1)}{2}} .$$

$S \circlearrowright L_\lambda$ where H acts diagonally & E, F act locally nilpotently.

$S \curvearrowright L_\lambda$ where H acts diagonally & E, F act locally nilpotently.

<u>Lemma</u>: $S \cdot v_r = (-1)^{\lambda - r} q^{(\lambda - r)(r+1)} v_{\lambda - r}$

Automorphism of $U_\hbar(sl_2)$

$$T: \begin{cases} H \longrightarrow -H \\ E \longrightarrow -FK \\ F \longrightarrow -K^{-1}E \end{cases}$$

$\forall u \in U_\hbar(sl_2)$, $v \in L_\lambda$, "$T(x) = S x S^{-1}$"

$$S(uv) = T(u)(Sv)$$

<u>Higher rank case</u>: any g

we get $T_i : U_\hbar(g) \longrightarrow U_\hbar(g)$ $\forall i \in I$,

T_i : $H_i \longmapsto -H_i$, $E_i \mapsto -F_i K_i$, $F_i \longmapsto -K_i^{-1} E_i$

for $i \neq j$,

$$T_i(F_j) = \sum^{-a_{ij}} (-1)^s q_i^s F_i^{(s)} F_j F_i^{(-a_{ij} - s)}$$

$$T_i(F_j) = \sum_{s=0}^{-a_{ij}} (-1)^s q_i^s F_i^{(s)} F_j F_i^{(-a_{ij}-s)}.$$

Similarly for E_j.

$$T_i(H) = s_i(H) \quad \forall\, H \in \mathfrak{h}.$$

<u>Theorems of Lusztig:</u>

$\{T_i\}_{i\in I}$ satisfy braid rel'ns:

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$$

Lusztig Intro to QG's.