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Friday, November 22, 2019 13:50

Let V be a f.d. v.s. over $\mathbb{C}$. $\{v_1, \dots, v_d\}$ basis.

Let $C = \mathrm{sym}(V)$

$$\mathrm{Sym}(V) = \bigoplus_{n=0}^{\infty} \underbrace{S^n V}_{\text{basis } \{v_1^{\alpha_1} \cdots v_d^{\alpha_d} \mid \sum \alpha_i = n\}}, \quad S^0 V = \mathbb{C}.$$

- $\Delta : C \longrightarrow C \otimes C$
  $1 \longmapsto 1 \otimes 1$
  $v \longmapsto v \otimes 1 + 1 \otimes v$

- $\varepsilon : C \longrightarrow \mathbb{C}$ counit.

Recall the cobar complex $(T^\bullet C, \delta)$

$$\begin{array}{ccccccccc}
C & \xrightarrow{0} & C & \xrightarrow{\delta} & C \otimes C & \xrightarrow{\delta} & C \otimes C \otimes C & \xrightarrow{\delta} & \cdots \\
\| & & \| & & \| & & \| & & \\
T^0 C & & T^1 C & & T^2 C & & T^3 C & &
\end{array}$$

$$\delta(a_1 \otimes \cdots a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n + \sum_{i=1}^{n} (-1)^i \cdots \otimes \Delta(a_i) \otimes \cdots + (-1)^{n+1} a_1 \otimes \cdots \otimes a_n \otimes 1,$$

$$\Lambda^\bullet V = \bigoplus_{n=0}^{\infty} \Lambda^n V$$

$\underline{\text{Ex}}$: $p \in C$ $\quad \delta(p) = 1 \otimes p - \Delta(p) + p \otimes 1 = 0$

$$\Leftrightarrow \Delta(p) = p \otimes 1 + 1 \otimes p$$

$$\Leftrightarrow p \in \underset{\underset{\Lambda^1 V.}{\|}}{V} \subset C.$$

$\underline{\text{Ex}}$ $\quad \delta(v_1 \otimes v_2) = 1 \otimes v_1 \otimes v_2 - (v_1 \otimes 1 + 1 \otimes v_1) \otimes v_2 + v_1 \otimes (v_2 \otimes 1 + 1 \otimes v_2) - v_1 \otimes v_2 \otimes 1 = 0$

$$\mu : T^n C \longrightarrow \Lambda^n V$$

use $pr_V : C \longrightarrow V$

$$\mu : \xi_1 \otimes \cdots \otimes \xi_n \longmapsto pr_V(\xi_1) \wedge \cdots \wedge pr_V(\xi_n)$$

$\alpha : \Lambda^n V \longrightarrow T^n C$ "skew symmetrization".

$\underline{\text{Theorem}}$ (1) $\mu : (T^\bullet C, \delta) \longrightarrow (\Lambda^\bullet V, 0)$ is a chain map.

(2) $\delta \circ \alpha = 0$, $\mu \circ \alpha(\omega) = n!\,\omega$ $\forall \omega \in \Lambda^n V$.

$$\begin{tikzcd} \Lambda^n V \arrow[r, "\alpha"] \arrow[rr, bend right, "\cong"'] & \mathrm{Ker}(\delta : T^n \to T^{n+1}) \arrow[r, two heads] & H^n(T^\bullet C, \delta) \end{tikzcd}$$

$\underline{\text{Dual Statement}}$ $\quad W = V^*$, $\quad A = \mathrm{Sym}(W)$

$$\rightsquigarrow (T^\bullet A, d), \quad d : T^n A \longrightarrow T^{n-1} A$$

$$\begin{aligned} a_1 \otimes \cdots \otimes a_n \longmapsto\ & \varepsilon(a_1)\, a_2 \otimes \cdots \otimes a_n \\ & + \sum_1^{n-1} (-1)^i \cdots a_i a_{i+1} \cdots \\ & + (-1)^n a_1 \otimes \cdots \otimes a_{n-1}\, \varepsilon(a_n). \end{aligned}$$

μ, α as before

Same statement.