11/1

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Data of finite root system

- Cartan matrix $A = (a_{ij})_{ij \in I}$ ← finite set

$a_{ii} = 2 \quad \forall\, i \in I$

$a_{ij} \in \mathbb{Z}_{\leq 0} \quad \forall\, i \neq j$

$\exists\, d_i \in \mathbb{Z}_{>0}$ s.t. $d_i a_{ij} = d_j a_{ji} \quad \forall\, i,j$, DA pos. def.

$\mathfrak{h} = \mathbb{C}$-span of $\{h_i\}$ $\qquad (h_i, h_j)_0 = \frac{1}{d_i} a_{ij}$

$\mathfrak{h}^* \ni \alpha_j \qquad \alpha_i(h_j) = a_{ij} \quad \forall\, i,j.$

$\tilde{\mathcal{U}}_\hbar$: algebra → over $\mathbb{C}[\![\hbar]\!]$ gen by

$h \in \mathfrak{h}$, $\{E_i, F_i\}_{i \in I}$.

<u>Rel^n's</u> :

$[h, h'] = 0$

$[h, E_i] = \alpha_i(h) E_i$

$[h, F_i] = -\alpha_i(h) F_i$.

$$[E_i, F_j] = \delta_{ij} \, \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$\left(q = e^{\hbar/2}, \; q_i = q^{d_i}, \; K_i = q_i^{h_i} \right)$$

- Δ, S, ε $\quad \left(\forall i, \; \{E_i, F_i, K_i\} \cong U_{d_i \hbar}(sl_2) \overset{\text{Hopf subalg}}{\subset} \mathcal{U} \right).$

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \text{et cetera}$$

- Need a non-degenerate form.

$\longrightarrow$ Define $(\cdot, \cdot)$ and mod out by the radical.

$\tilde{\mathcal{U}}^{\leq 0}$ - gen by $\mathfrak{h}, \{F_i\}$

$\tilde{\mathcal{U}}^{\geq 0}$ - gen by $\mathfrak{h}, \{E_i\}$

(Hopf subalgebras)

$$\tilde{\mathcal{U}}^{\leq 0} \times \tilde{\mathcal{U}}^{\geq 0} \longrightarrow \mathbb{C}((\hbar))$$

- $(1, x) = \varepsilon(x)$ $\quad\Big|\quad$ $(y, x_1 x_2) = (\Delta(y), x_1 \otimes x_2)$

$(y, 1) = \varepsilon(y)$ $\quad\Big|\quad$ $(y_1 y_2, x) = (y_1 \otimes y_2, \Delta^{op}(x))$

$$(h, h') = \frac{(h, h')_0}{\ln(q)}$$

$$(h, E_i) = 0 = (F_i, h)$$

$$(F_i, E_j) = \frac{\delta_{ij}}{q_i - q_i^{-1}}$$

$$(\cdot, \cdot) : \tilde{U}^{\leq 0} \times \tilde{U}^{\geq 0} \longrightarrow \mathbb{C}((\hbar)) .$$

<u>Properties of $(\cdot,\cdot)$:</u>

(1) $U^0 \cdot U^- \times U^0 \cdot U^+$

(U^- = free assoc alg on $\{F_i\}$, $\deg(F_i) = -\alpha_i$
U^+ = " " $\{E_i\}$ $\deg(E_i) = \alpha_i$

$$(P_1 \cdot y, P_2 \cdot x) = (P_1, P_2)(y, x).$$

$$U^+ = \bigoplus_{\mu \in Q_+} U^+_\mu$$

each graded piece is f.d.

(2) on $U^0 \times U^0$,

the form is non-degenerate.

let $\{x_i\}_{i \in I}$ be an o.n.b. of $\mathfrak{h}$, $(\cdot,\cdot)_0$.

$$\left(\prod_i x_i^{n_i}, \prod_i x_i^{m_i}\right) = \delta_{\underline{n}, \underline{m}} \frac{\prod n_i!}{\ln(q)^{\sum_i n_i}}$$

$\rightsquigarrow$ canonical tensor $= q^{\sum_i x_i \otimes x_i}$ $(= R_0)$

(3) $(U^-_{-\mu}, U^+_{+\nu}) = 0$ if $\mu \neq \nu$

$\mathrm{rad}^{\geq 0} \subset \mathcal{U}^{\geq 0}$ consists of x s.t. $(y, x) = 0 \ \forall \ y \in \mathcal{U}^{\leq 0}$.