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Simple Lie algebras & repns

$\mathfrak{g}(A) \leftsquigarrow A = (a_{ij})_{i,j \in I}$ Cartan Matrix

Gens: $\mathfrak{h}, \{e_i, f_i\}_{i \in I}$

Rel^{n}:

① $\mathfrak{h}$ is abelian

② $ad(h) \cdot e_i = \alpha_i(h) \cdot e_i$

$ad(h) f_i = -\alpha_i(h) \cdot f_i$

③ $[e_i, f_j] = \delta_{ij} h_i$

④ $(ad(e_i))^{1-a_{ij}} e_j = 0 = (ad f_i))^{1-a_{ij}} f_j$ for $i \neq j$

Rk: (1) $\forall i \in I, \{e_i, f_i, h_i\} \cong sl_2$

(2) $\forall i \neq j$

$$sl_2^{(i)} \circlearrowright \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{g}_{\alpha_j + k\alpha_i}$$

1-d: $\mathbb{C}(e_j + k e_i)$

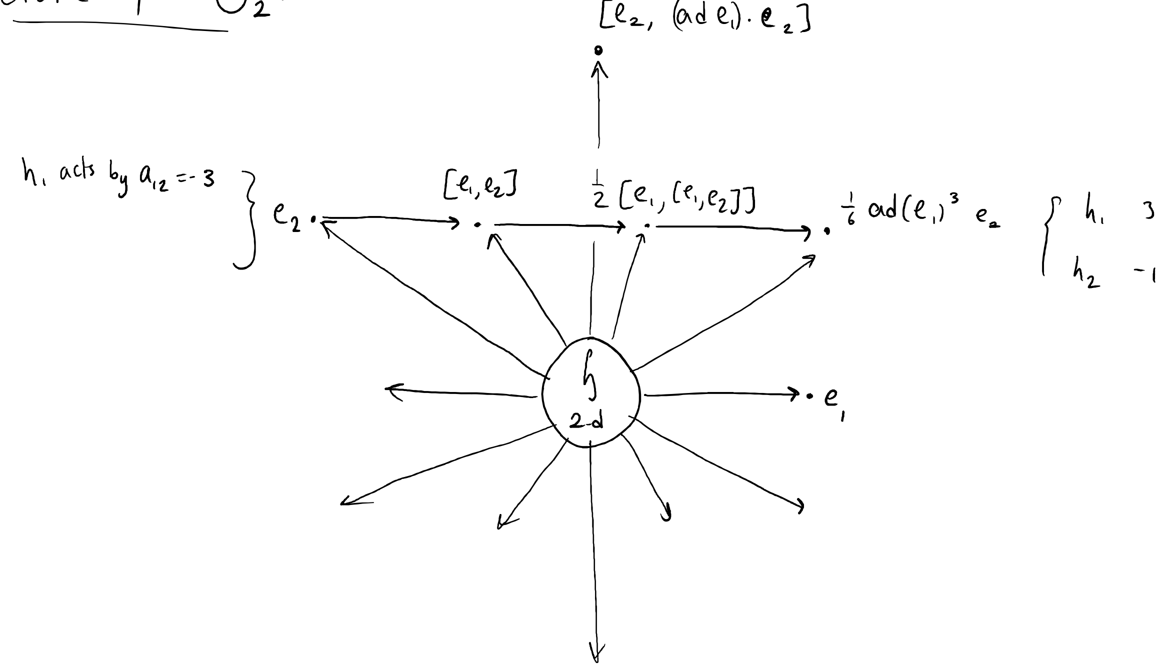
is $\cong L_{-a_{ij}}$

$$\bar{s}_i = \exp(\mathrm{ad}\, e_i)\, \exp(-\mathrm{ad}\, f_i)\, \exp(\mathrm{ad}\, e_i)$$

acts on $\mathfrak{g}$

$$\bar{s}_i : \mathfrak{g}_\alpha \longrightarrow \mathfrak{g}_{s_i(\alpha)}$$

Picture of G_2:



$\{\bar{s}_i\}$ satisfy braid rel$^{\underline{n}}$s:

$$\underbrace{\bar{s}_i \bar{s}_j \bar{s}_i \cdots}_{m_{ij}} = \underbrace{\bar{s}_j \bar{s}_i \bar{s}_j \cdots}_{m_{ij}}$$

Tits Theorem

Finite-dim'l repns of $\mathfrak{g}$

$\mathfrak{g} \circlearrowright V$ f.d. v.s. over $\mathbb{C}$.

(i) $\mathfrak{h} \subset \mathfrak{g}$ acts semisimply (this is by sl_2-repn thy)

<u>Notations</u>: $\gamma \in \mathfrak{h}^* \rightsquigarrow V[\gamma] = \{ v \in V \mid h \cdot v \;\; \gamma(v) \;\; \forall h \in \mathfrak{h} \}$

γ-weight space

(sl_2-rep theory $\Longrightarrow \gamma(h_i) \in \mathbb{Z} \;\; \forall i \in I$)

Weight Lattice

$$P = \{ \gamma \in \mathfrak{h}^* \mid \gamma(h_i) \in \mathbb{Z} \;\; \forall i \in I \}$$

$$= \bigoplus_{i \in I} \mathbb{Z} \varpi_i \qquad \text{where} \quad \left\{ \begin{matrix} \varpi_i \in \mathfrak{h}^* \\ \varpi_i(h_j) = \delta_{ij} \end{matrix} \right\} \text{ fund. weights.}$$

$$V = \bigoplus_{\mu \in P} V[\mu]$$

(ii) $\forall i \in I$, $\quad e_i : V[\mu] \longrightarrow V[\mu + \alpha_i]$

$\qquad\qquad f_i : V[\mu] \longrightarrow V[\mu - \alpha_i]$

sublattice

$$P \supset Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$$

$$\cup$$

$$Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$$

($|P/Q|$ finite $= n+1$ for type A_n)

$$Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$$

$$\left(\begin{array}{l} \overset{\shortparallel}{n+1} \text{ for type } A_n \\ \underline{\text{ex}} \ \det \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}_{n \times n} = n+1 \end{array} \right)$$

Now assume V is irreducible

(1) choose $\lambda \in \mathfrak{h}^*$ s.t. $V[\lambda] \neq 0$

$$V[\lambda + \alpha_i] = 0 \quad \forall i \in I$$

(2) choose $v \in V[\lambda]$; $v \neq 0$

$$e_i v = 0 \quad \forall \ i \in I$$

$$h v = \lambda(h) v \qquad \forall h \in \mathfrak{h}$$

$$\left(\begin{array}{l} \mathfrak{sl}_2\text{-rep thy} \implies \lambda(h_i) \in \mathbb{Z}_{\geq 0} \ \forall \ i \in I \\ \qquad\qquad \& \ f_i^{\lambda(h_i)+1} v = 0 \end{array} \right)$$

$V' :=$ (sub-repn gen by v)

$= \mathbb{C}$ - span of $\{ f_{i_r} \cdots f_{i_2} \cdot f_{i_1} \}_{\substack{r \geq 0 \\ i_1, \dots, i_r \in I}} \subset V$

by irreducibility, $V' = V$

Cor: $V[\lambda]$ is 1-dim. $v \in V$ is (up to a scaler) unique weight vector s.t. $e_i v = 0 \ \forall i \in I$.

$$\begin{matrix} P_+ \\ \psi \\ \lambda \end{matrix} \rightsquigarrow \text{irred. f.d. repn of } \mathfrak{g}$$

$$V \ni v \quad \text{s.t.} \begin{cases} e_i v = 0 \ \forall i \\ hv = \lambda(h) v \ \forall h \in \mathfrak{h} \\ f_i^{\lambda(h_i)+1} v = 0 \ \forall i \end{cases}$$

highest-weight vector

denote it by L_λ

"Weyl group" action on V

Theorem (Harish-Chandra): L_λ is f.d. and irreducible.

Precisely, $\overline{s_i} : V \longrightarrow V$ v.s. isomorphism

$$\overline{s_i} : V[\mu] \xrightarrow{\sim} V[s_i(\mu)]$$

$$P(V) = \{ \gamma \in P \mid V(\gamma] \neq 0 \}$$

$P(V) \cap P_+$ is finite.

Any wt space is in "W"-orbit of $P[V] \cap P_+$.

Complete Reducibility holds

Casimir element (recall for sl_2 $C := \frac{h^2}{2} + ef - fe$
$= \frac{h^2}{2} + h + 2fe$)

1. There's a symmetric nondegenerate bilinear form

$$(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$$

s.t.

(i) $(\cdot,\cdot)\big|_{\mathfrak{h}\times\mathfrak{h}}$ same as before

(ii) $(e_i, f_i) := \frac{\delta_{ij}}{d_i}$ (where $d_i = \frac{(\alpha_i,\alpha_i)}{2}$)

(iii) $([x,y],z) = (x,[y,z])$ $\forall x,y,z \in \mathfrak{g}$.

$$\bigoplus_{\alpha\in R_+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in R_+} \mathfrak{g}_{\alpha}$$

$(\cdot,\cdot)\big|_{\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}}$ is non-degen

$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$.

Choose $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ s.t. $(e_\alpha, f_\alpha) = \frac{1}{d_\alpha}$

$\forall \alpha \in R_+$

Then $C = C^\circ + \sum_{\alpha \in R_+} d_\alpha (e_\alpha f_\alpha + f_\alpha e_\alpha)$

(quadratic) element in terms of $h \in \mathfrak{h}$.

<u>Fact</u> V f.d. v.s., $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{C}$ nondegen form

$\rightsquigarrow$ Canonical tensor of $(\cdot,\cdot)$ is an element of $V \otimes V$

- $\{v_i\}$ basis of V $\rightsquigarrow$ $\{v^i\}$ dual to $\{v_i\}$ wrt $(\cdot,\cdot)$

- $\sum_{i=1}^{\dim V} v^i \otimes v_i \in V \otimes V$ is indep of the basis.

$$\begin{array}{ccc} \sum_{i=1}^{\dim V} v^i \otimes v_i & \in & V \otimes V \\ \downarrow & & \downarrow \wr \\ \mathrm{Id}_V \in \mathrm{End}\, V & \cong & V^* \otimes V \end{array}$$

(Hom-$\otimes$ adjointness)

$$C^\circ v_\lambda = (\lambda, \lambda)\, v_\lambda$$

($v_\lambda \in L_\lambda$ weight vector)

$\rightarrow$ C is central

$\rightarrow$ $$C\Big|_{L_\lambda} = (\lambda + 2\rho, \lambda)\, Id_{L_\lambda}$$

$\mathfrak{g} \circlearrowright V$ f.d. repn

$$V = \bigoplus_{\mu \in P} V[\mu]$$

<u>Character of V</u>

$$\chi_V := \sum_{\mu \in P} (\dim V[\mu])\, e^\mu$$

is W-invariant

$e^\mu \in \mathbb{Z}[e^{\pm \omega_i} : i \in I]$

$\circlearrowleft$

W

- Weyl Character Formula

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} (1-e^{-\alpha})}$$

eg $\mathfrak{g} = sl_2$ $\quad L_n \qquad n \in \mathbb{Z}_{\geq 0} \longleftrightarrow P_+$

$W \cong \mathbb{Z}/_2 = \langle \sigma \rangle$

$$\chi_{L_n} = \frac{e^{n\frac{\alpha}{2}} - e^{(-n-2)\frac{\alpha}{2}}}{1-e^{-\alpha}} \qquad e^{\alpha/2} = z$$

$$= \frac{z^n - z^{-n-2}}{1-z^{-2}}$$

$$= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}} = z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n}.$$