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Monday, October 7, 2019 13:50

Construction of simple Lie algebras

$R \subset E^* \setminus \{0\}$ root system $\left\{ \begin{array}{l} \{\alpha_i\}_{i \in I} \text{ simple roots} \\ \nu : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h} \qquad h_i = \alpha_i^\vee = \dfrac{2\nu(\alpha_i)}{(\alpha_i, \alpha_i)} \end{array} \right\}$

$\rightsquigarrow \tilde{\mathfrak{g}}$: Lie Algebra

- generators : $\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C}$,

  $\{e_i, f_i\}_{i \in I}$

- relations : • $\mathfrak{h}$ is abelian,

  • $\mathrm{ad}(h)\, e_i = \alpha_i(h)\, e_i$

  $\mathrm{ad}(h)\, f_i = -\alpha_i(h)\, f_i$ $\quad \forall i \in I$

  • $[e_i, f_j] = \delta_{ij}\, h_i$

<u>Prop</u> $\exists!$ max'l ideal $\tilde{r} \subsetneq \tilde{\mathfrak{g}}$

$\rightsquigarrow \mathfrak{g} := \tilde{\mathfrak{g}} / \tilde{r}$ simple lie alg.

$$\tilde{\mathfrak{g}} = \underbrace{\tilde{n}_-}_{\text{gen by } \{f_i\}_{i \in I}} \oplus \mathfrak{h} \oplus \underbrace{\tilde{n}_+}_{\text{gen by } \{e_i\}_{i \in I}}$$ triangular decomposition.

root spaces

finer $\tilde{\mathfrak{g}}_\gamma = \{ x \in \tilde{\mathfrak{g}} \mid [h, x] = \gamma(h)\, x \quad \forall h \in \mathfrak{h} \}$

$(\forall \gamma \in \mathfrak{h}^*)$

$$\tilde{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \tilde{\mathfrak{g}}_{\pm\alpha} \qquad \left(Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathfrak{h}^* \right)$$

If $\mathfrak{a} \subseteq \tilde{\mathfrak{g}}$ is an ideal then

$$\mathfrak{a} = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \tilde{\mathfrak{g}}_{\pm\alpha} \cap \mathfrak{a}$$ is a direct sum of ideals.

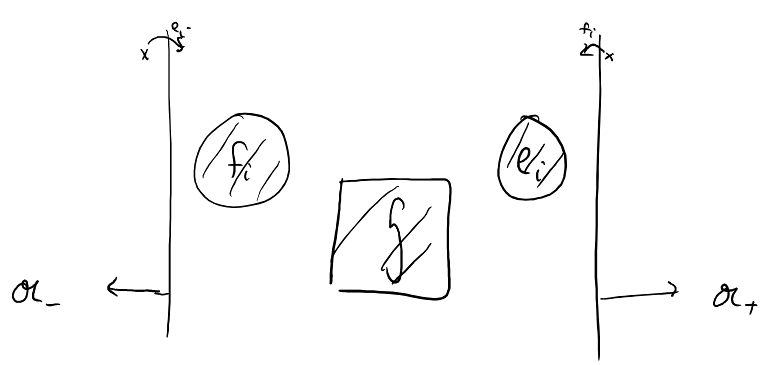
$\tilde{\mathfrak{g}}_\alpha$ is finite dimensional,

$$\dim \tilde{\mathfrak{g}}_\alpha = \dim \tilde{\mathfrak{g}}_{-\alpha}$$

$$\tilde{\mathfrak{g}}_{\pm\alpha_i} = \begin{cases} \mathbb{C}\, e_i & \text{for } + \\ \mathbb{C}\, f_i & \text{for } - \end{cases}$$

$$\tilde{\mathfrak{g}}_{k\alpha_i} = 0 \quad \text{for all } k \in \mathbb{Z} \text{ s.t. } |k| \geq 2.$$

(proper)
ideals in $\tilde{\mathfrak{g}}$ $\quad \mathfrak{a} \subsetneq \tilde{\mathfrak{g}} \rightsquigarrow \mathfrak{a}_\pm = \mathfrak{a} \cap \tilde{\mathfrak{n}}_\pm$



Remark

(i) $\mathfrak{a}_\pm$ are ideals $\qquad \mathfrak{a} = \mathfrak{a}_- \oplus \mathfrak{a}_+$.

$$[e_i, \mathfrak{a}_+] \subset \mathfrak{a}_+$$

$$[f_i, \mathfrak{a}] \subset \mathfrak{a}_+$$

(ii) if $x \in \mathfrak{a}_\alpha$ and α is of smallest height s.t. $\mathfrak{a}_\alpha \neq 0$,

Then $[f_i, x] = 0 \ \forall \ i \in I$

(iii) Conversely if $x \in \tilde{\mathfrak{n}}_+$ is s.t.

$$[f_i, x] = 0 \qquad \forall i \in I,$$

Then ideal gen by x is again in $\tilde{\mathfrak{n}}_+$

(Same for $-$ and e_i)

Define $\forall i \neq j \in I$; $\quad \theta_{ij}^+ = \mathrm{ad}(e_i)^{1-a_{ij}} e_j \in \tilde{\mathfrak{n}}_+$

$$\theta_{ij}^- = \mathrm{ad}(f_i)^{1-a_{ij}} f_j \in \tilde{\mathfrak{n}}_-$$

<u>Lemma</u> $\forall k \in I$,

$$[e_k, \theta_{ij}^-] = 0$$

$$[f_k, \theta_{ij}^+] = 0$$

Pf $\cdot$ <u>$k \neq i, j$</u> this is clear. $\left(\begin{array}{l} \mathrm{ad}(e_k) \cdot f_i = 0 \\ \qquad\qquad f_j = 0 \end{array} \right)$

• $\underline{k=j}$ $\quad (\mathrm{ad}\, e_i)\cdot(\mathrm{ad} f_j)^{1-a_{ij}} f_j = (\mathrm{ad} f_i)^{1-a_{ij}} h_j = 0$ if $a_{ij} \le -1$

If $a_{ij}=0$, then $a_{ji}=0$; and

$$(\mathrm{ad} f_i)^1 h_j = [f_i, h_j] = \cancelto{0}{a_{ij}} f_i = 0$$

• $\underline{k=i}$ $\quad \mathrm{ad}(e_i)\cdot f_j = 0$

$\mathrm{ad}(h_j)\cdot f_j = -a_{ij} f_j$

$\rightsquigarrow \mathrm{ad}(e_i)\cdot\left(\mathrm{ad}(f_i)^{1-a_{ij}}\cdot f_j\right) = 0$ by

$ef^k = f^k e + k f^{k-1}(h-k+1)$

$\square$

$$\begin{array}{ccccc} & & \tilde{r} \subsetneq \tilde{\mathfrak{g}} & & \\ & & \parallel & & \\ & \tilde{\mathfrak{n}}_- & \oplus\ \mathfrak{h}\ \oplus & \tilde{\mathfrak{n}}_+ & \\ & \cup_{\!\times} & & {}_{\times}\!\cup & \\ \tilde{r}_- & & & & \tilde{r}_+ \end{array}$$

$$\begin{array}{c|ccc|c} \theta^-_{ij} & \boxed{\begin{matrix} f_i \\ i\in I\end{matrix}} & \boxed{\mathfrak{h}} & \boxed{\begin{matrix} e_i \\ i\in I\end{matrix}} & \theta^+_{ij} \\ (i\neq j) & & & & (i\neq j) \end{array}$$

"$\underline{\text{Weyl Group Action}}$":

$$\bar{\mathfrak{g}} = \tilde{\mathfrak{g}} \big/ \langle \theta^{\pm}_{ij} : i\neq j \rangle$$

$$= \bar{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{n}}_-$$

$\circlearrowright$

$\mathfrak{sl}_2^i$ acts locally nilpotently

$\|$

$\langle e_i, f_i, h_i \rangle$

$\Rightarrow$ we have $s_i \circlearrowright \tilde{\mathfrak{g}}$

$$s_i = \exp(\mathrm{ad}\, e_i)\, \exp(-\mathrm{ad} f_i)\, \exp(\mathrm{ad}\, e_i)$$

Remark $s = \exp(e) \exp(-f) \exp(e)$ makes sense on any $\mathfrak{sl}_2$-representation where e, f act locally nilpotently.

Defn $X \in \mathrm{End}(V)$ acts locally nilpotently if $\forall v \in V,\ \exists n \in \mathbb{N}$ s.t. $X^n \cdot v = 0$.

Thus $s_i : \tilde{\mathfrak{g}}_\alpha \xrightarrow{\sim} \tilde{\mathfrak{g}}_{s_i(\alpha)}$ $\forall i \in I$. (on friday)

isomorphism of vector spaces

Prop $\tilde{r}_\pm = \langle \theta_{ij}^\pm \mid i \neq j \in I \rangle$

$\|$

$\tilde{r}_\pm \supseteq r_\pm$

" . . . " $\curvearrowleft$ $\tilde{\mathfrak{g}} \xrightarrow{\pi} \tilde{\mathfrak{n}}$ / ... assume height (α) is minimum

$$\text{"}W\text{"} \circlearrowright \bar{\mathfrak{g}} \xrightarrow{\pi} \tilde{\mathfrak{g}}/\tilde{r}$$

assume height (α) is minimum s.t. $\mathrm{Ker}(\pi)_\alpha \neq 0$

$\mathrm{Ker}(\pi) = \tilde{r}/r$. if non-zero, let $\kappa \in \mathrm{Ker}(\pi)$.

$\kappa \in \tilde{r}_{+\alpha} \quad (\alpha \in Q_+ \setminus \{0\})$

Then $s_i(\kappa) \in \tilde{r}_+ \rightsquigarrow s_i(\alpha)$ has larger ht than α.

$$s_i(\alpha) = \alpha - \alpha(h_i)\,\alpha_i$$

so $\quad (\alpha, \alpha_i) \leq 0 \quad \forall i.$

$$\alpha \in Q_+ \Rightarrow \alpha = \sum_{i\in I} n_i \alpha_i \qquad n_i \geq 0$$

$$(\alpha, \alpha) = \sum_{i \in I} n_i (\alpha, \alpha_i) \leq 0 \Rightarrow \alpha = 0, \text{ contradiction!}$$

$(\cdot, \cdot)$ positive def.

So $\tilde{r} = r$. $\square$

Similar proof gives the following:

$\mathfrak{g}_\alpha \neq (0) \Rightarrow \alpha$ is W-conjugate to some $\alpha_i \quad (i \in I)$.

$$\mathfrak{g}^+ = \bigoplus_{\alpha \in Q_+ \cap \left(\bigcup_{i\in I} W\alpha_i\right)} \mathfrak{g}_\alpha \subsetneq n_+$$

<u>Cor.</u> $\dim \mathfrak{g} = |I| + |R|$

$$= |I| + 2\,|R_+|$$

$$\mathfrak{g}_\alpha \neq 0 \underset{\alpha = w(\alpha_i)}{\Longrightarrow} \mathfrak{g}_\alpha \xrightarrow{\sim} \mathfrak{g}_{\alpha_i}$$

$$\Longrightarrow \dim \mathfrak{g}_\alpha = 1.$$

So $$\mathfrak{g} = \left(\bigoplus_{\alpha \in R_+} \mathfrak{g}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \right).$$

Summary: Input $A = (a_{ij})_{i,j \in I}$ (Cartan matrix of a root system).

$\rightsquigarrow$

Output - a f.d simple Lie alg $\mathfrak{g}(A)$:

gen's	relns	
$\{h_i, e_i, f_i\}_{i \in I}$	$[h_i, h_j] = 0$	$[e_i, f_j] = \delta_{ij} h_i$
	$[h_i, e_j] = a_{ij} e_j$	$ad(e_i)^{1-a_{ij}} e_j = 0$
	$[h_i, f_j] = -a_{ij} f_j$	$ad(f_i)^{1-a_{ij}} f_j = 0$

Classification Theorem (E. Cartan, Weyl, Chevalley, Killing)

Every f.d. simple Lie algebra arises this way.

Example 1 A_2 $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \longleftarrow R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

$h_1 \quad e_1 \quad f_1$

$\rightsquigarrow n_+$ sub alg gen by e_1, e_2.

$h_2 \quad e_2 \quad f_2$

$$\begin{matrix} e_1 \\ e_2 \end{matrix} \rightsquigarrow [e_1, e_2] \rightsquigarrow [e_1, [e_1, e_2]]$$

$$[e_1, e_2] = e_3$$

$$[e_1, [e_1, e_2]] = (ad(e_1))^2 e_2 = 0$$

$$n_+ = \text{span}\{e_1, e_2, e_3\}$$

$$\rightsquigarrow \underset{(n)}{\mathfrak{g}(A_2)} \cong \underset{(n+1)}{\mathfrak{sl}_3} \quad (\underset{(n+1)\times(n+1)}{3\times 3} \text{ matrices } \overset{\text{over } \mathbb{C}}{\vee} \text{ of trace } 0)$$

$$h_1 \longleftrightarrow \begin{bmatrix} -1 & & \\ & -1 & \\ & & 0 \end{bmatrix} \qquad e_1 \longleftrightarrow \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \\ & 0 \end{bmatrix} \quad (= f_1^T)$$

$$h_2 \longleftrightarrow \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix} \qquad e_2 \longleftrightarrow \begin{bmatrix} 0 & \\ & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{bmatrix} \quad (= f_2^T)$$

$$e_3 = \begin{bmatrix} 0 & 0 & 1 \\ \multicolumn{2}{c}{\multirow{2}{*}{0}} & 0 \\ & & 0 \end{bmatrix} = [e_1, e_2]$$