10/4

Friday, October 4, 2019 13:59

$sl_2(\mathbb{C})$

- $\mathrm{Rep}_{fd}(sl_2)$ is a semisimple category
- $\mathrm{Irred}_{fd}(sl_2) \longleftrightarrow \mathbb{Z}_{\geq 0}$

$sl_2 \circlearrowright V$ f.d $\rightsquigarrow$ $s \in GL(V)$

$s = \exp(e)\exp(-f)\exp(e)$

eg $V = \mathbb{C}^2$; $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

For $k \in \mathbb{Z}$; $s : V[k] \xrightarrow{\sim} V[-k]$. - $(*)$

<u>Proof of $(*)$</u>

$sl_2 \circlearrowright V$, $v \in V$, $h \cdot v = kv$

(T.S.) $h \cdot (s(v)) = -k\, s(v)$

equivalently, $\boxed{s \cdot h \cdot s^{-1} = -h}$

$$\boxed{\exp(a) \cdot b \cdot \exp(-a) = \exp(\mathrm{ad}(a)) \cdot b}$$

$$\left(1 + a + \frac{a^2}{2!} + \cdots\right) b \left(1 - a + \frac{a^2}{2} - \cdots\right) = b + [a,b] + \cdots$$

$$\begin{cases} \mathrm{Ad}(A)\cdot X = AXA^{-1} \\ \mathrm{Ad}(\exp(a))\cdot b = \exp(\mathrm{ad}(a))\cdot b \end{cases}$$

$$shs^{-1} = \exp(\mathrm{ad}(e))\exp(-\mathrm{ad}(f))\underbrace{\exp(\mathrm{ad}(e))\cdot h}$$

$$\downarrow$$

$$h + [e,h] + \cancel{\frac{[e,[e,h]]}{2}}^{\,0} + \cdots \quad \text{all } 0$$

$$\| $$

$$h-2e$$

$$\exp(-\mathrm{ad}(f))\cdot(h-2e)$$

$$= h - [f,h] + \cdots \quad \text{all } 0$$

$$-2\left(e - [f,e] + \frac{[f,[f,e]]}{2!} - \cancel{\frac{(\mathrm{ad}f)^3\cdot e}{3!}}^{\,0} + \cdots\right) \quad \text{all } 0$$

$$= h - \cancel{2f} - 2(e + h - \cancel{f})$$

$$= -h - 2e$$

$$\exp(\mathrm{ad}(e))\cdot(-h-2e)$$

$$= -(h-2e) - 2e = -h$$

$\square$

eg $V = sl_2 \circlearrowleft sl_2$ $\quad s: \begin{cases} e \longmapsto -f \\ f \longmapsto -e \\ h \longmapsto -h \end{cases}$

Trick to compute $s \circlearrowright L_n$ $(n \in \mathbb{Z}_{\geq 0})$

$$s\Big|_{\mathbb{C}^2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad s\Big|_{V_1 \otimes V_2} = s\Big|_{V_1} \otimes s\Big|_{V_2}$$

$$\rightsquigarrow s \circlearrowright \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n\text{-fold}} \leftarrow 2^n\text{-dim'l}$$

$\uparrow$ basis given by $\underline{a} = a_1 \cdots a_n$, each $a_i = \uparrow$ or $\downarrow$.

$$s : \begin{array}{l} \uparrow \longmapsto \downarrow \\ \downarrow \longmapsto \uparrow \end{array} \quad \text{comp. wise}$$

$$V[n-2k] \xrightarrow{(-1)^{n-k} \{\uparrow \leftrightarrow \downarrow\}} V[-n+2k]$$

$\uparrow$

basis $\underline{a} \,\Big|\, \#\{j \mid a_j = \downarrow\} = k$, $\binom{n}{k}$-dim'l

L_n = subrepn gen by $\uparrow\uparrow \cdots \uparrow \in (\mathbb{C}^2)^{\otimes n}$

$$\rightsquigarrow s(v_j) = (-1)^{n-j} v_{n-j} \qquad 0 \leq j \leq n.$$

Constructing simple lie algebras

Constructing simple lie algebras

Defn Let $\mathfrak{g}$ be a lie algebra.

An <u>ideal</u> $\mathfrak{a} \subset \mathfrak{g}$ is a subspace s.t.

$$\left.\begin{matrix} x \in \mathfrak{g} \\ a \in \mathfrak{a} \end{matrix}\right\} \Rightarrow [x, a] \in \mathfrak{a}$$

We say $\mathfrak{g}$ is simple if (0) and $\mathfrak{g}$ are the only ideals.

Convention: 1-dim'l lie alg is not considered simple.

C. Chevalley (1948) - Sur la Classification des algèbres de Lie et de leur représentation.

Kac - inf. dim'l lie algebras Ch 1.

R: root system (irreducible) $\rightsquigarrow$ $\mathfrak{g}$ simple lie alg.

- $\mathfrak{h} = E \otimes_{\mathbb{R}} \mathbb{C}$

$\tilde{\mathfrak{a}} =$ [illegible]

$\tilde{\mathfrak{g}}$ = Lie algebra generated by $\mathfrak{h}$, $\{e_i, f_i\}_{i \in I}$

Rel$^{\underline{n}}$s:

- $\mathfrak{h}$ is abelian $\quad ([h_1, h_2] = 0 \ \ \forall h_1, h_2 \in \mathfrak{h})$.
- $[h, e_i] = \alpha_i(h) e_i$
  $[h, f_i] = -\alpha_i(h) f_i$ $\quad \forall i \in I, \ h \in \mathfrak{h}$
- $[e_i, f_j] = \delta_{ij} h_i$ $\quad$ where $h_i = \alpha_i^\vee$.

Properties Triangular decomposition.

(1) $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$ $\quad$ (as vector spaces)

gen by f_i $\qquad$ gen by e_i

Let $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subseteq \mathfrak{h}^*$

Define $\forall \gamma \in \mathfrak{h}^*$

$$\tilde{\mathfrak{g}}_\gamma = \{ x \in \tilde{\mathfrak{g}} \mid [h, x] = \gamma(h) x \ \forall h \in \mathfrak{h} \}$$

Then (2) $\tilde{n}_\pm = \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \tilde{\mathfrak{g}}_{\pm\alpha}$, $\dim \tilde{\mathfrak{g}}_\alpha < \infty$

(3) we have an involution $\tilde{\omega}$: $e_i \longmapsto -f_i$
$f_i \longmapsto -e_i$
$h \longmapsto -h$

(4) $\tilde{n}_\pm$ are freely generated by $\{e_i\}_{i \in I}$, $\{f_i\}_{i \in I}$

Prop if $\mathfrak{a} \subsetneq \mathfrak{g}$ is a proper ideal then $\mathfrak{a} \cap \mathfrak{h} = (0)$.

(hence $\exists!$ maximal proper ideal $\tilde{r} \subsetneq \tilde{\mathfrak{g}}$)

Then $\mathfrak{g} = \tilde{\mathfrak{g}}/\tilde{r}$ is a simple Lie alg.

~~pf~~ if $h \in \mathfrak{h} \cap \mathfrak{a}$, $h \neq 0$, pick $i \in I$ s.t. $\alpha_i(h) \neq 0$.

$\Rightarrow [h, e_i] = \alpha_i(h) e_i$.

$\Rightarrow e_i, h_i, f_i \in \mathfrak{a}$

$\Rightarrow e_j, f_j, h_j \in \mathfrak{a}$ for $j \in I$ s.t. $a_{ij} \neq 0$.

Keep going, since R is irreducible the Dynkin diagram is connected, so $\{e_i, f_i, h_i\}_{i \in I} \subset \mathfrak{a} \Rightarrow \mathfrak{a} = \tilde{\mathfrak{g}}$. $\square$

Theorem For $i \neq j$, $i, j \in I$, let

$$\Theta_{ij}^{+} = (\mathrm{ad}(e_i))^{1-a_{ij}} \cdot e_j$$

$$\Theta_{ij}^{-} = (\mathrm{ad}(f_i))^{1-a_{ij}} f_{ij}$$

Then $\tilde{r} = \langle \Theta_{ij}^{\pm} \rangle_{\substack{i,j \in I \\ i \neq j}}$

This gives rel'ns that could have been used to define $\mathfrak{g}$.

(the last few corresponding to $\Theta_{ij}^{\pm} = 0$ are Serre rel^{n}s).

$\mathfrak{g} = \tilde{\mathfrak{g}} / \tilde{r}$ is automatically simple.

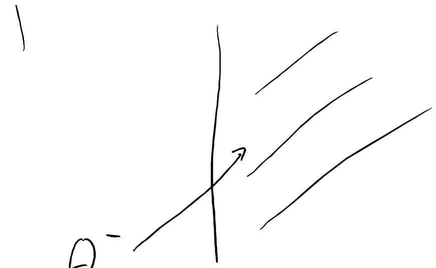
Idea for pf of Theorem:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$$

$\cup$

$\tilde{r}$ $\quad \tilde{r}_{\pm} = \tilde{r} \cap \tilde{\mathfrak{n}}$ $(i \in I)$ $\qquad \tilde{r}_-$

$\tilde{r} \qquad \tilde{r}_{\pm} = \tilde{r} \cap \tilde{n} \quad (i \in I)$



$\theta_{ij}^{-} \in \tilde{r}_{-}$

$\underline{\text{Step 1}}$ $\theta_{ij}^{\pm} \in \tilde{r}$

$$\theta_{ij}^{-} = (\operatorname{ad} f_i)^{1-a_{ij}} f_j$$

$\underline{\text{Claim}}$ $[e_k, \theta_{ij}^{-}] \quad \forall k \in I.$

$\underline{k=i}$:

$$\begin{array}{c} sl_2 \underset{\operatorname{ad}}{\circlearrowright} \mathfrak{g} \\ \uparrow \\ \{e_i, f_i, h_i\} \end{array}$$

$[e_i, f_j] = 0$

$[h_i, f_j] = -a_{ij} f_j$

$\Rightarrow \operatorname{ad}(e_i) \cdot \left((\operatorname{ad}(f_i))^{1-a_{ij}} f_j \right) = 0$

by sl_2-repn theory.