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$\mathcal{U} = \mathcal{U}_\hbar(sl_2)$ - Hopf algebra

- Algebra structure: Generators - $\{H, E, F\}$.

rel^n's :
$$[H, E] = 2E$$
$$[H, F] = -2F$$
$$[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}} = \frac{K - K^{-1}}{q - q^{-1}}$$

$q = e^{\hbar/2}$

$K = e^{\hbar H/2}$

- coproduct $\Delta : \mathcal{U} \longrightarrow \mathcal{U} \otimes \mathcal{U}$
$$\Delta(H) = H \otimes 1 + 1 \otimes H$$
$$\Delta(E) = E \otimes 1 + K \otimes E$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$$

Counit : $\varepsilon : \mathcal{U} \longrightarrow \mathbb{C}[[\hbar]]$

$E, F, H \longmapsto 0$

$$\left((n+1)\text{-dim'l } \mathbb{C}\text{-v.s.}\right) \underset{\mathbb{C}}{\otimes} \mathbb{C}[[\hbar]]$$
$$\|$$
L_n basis $v_0, \dots, v_n$

extend scalars

$$H \cdot v_k = (n-2k) v_k$$

$$E \cdot v_k = \underbrace{[n-k+1]_q}\, v_{k-1}, \quad F \cdot v_k = [k+1]_q\, v_{k+1}$$

$$E\cdot v_k = \underbrace{[n-k+1]_q}\, v_{k-1}\ , \quad F\cdot v_k = [k+1]_q\, v_{k+1}$$

$$[\ell]_q = \frac{q^{\ell}-q^{-\ell}}{q-q^{-1}} = q^{\ell-1}+q^{\ell-3}+\cdots+q^{-\ell+1} \longrightarrow \ell \quad \text{as } q\to 1$$

Gaussian Integers

(or $\hbar\to 0$)

<u>Lemma</u>: These give an action of $U_\hbar(sl_2)$ on L_n.

Pf Check : $[E,F] = \dfrac{K-K^{-1}}{q-q^{-1}}$.

$$E\cdot(F\cdot v_k) = [k+1]\; F\, v_{k+1} = [k+1][n-k]\cdot v_k$$

$$F\cdot(E\cdot v_k) = [n-k+1]\, E\, v_{k-1} = [n-k-1][k]\cdot v_k$$

$$[E,F]\, v_k = \underbrace{\left([k+1][n-k] - [n-k+1][k]\right)}\, v_k$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\parallel \leftarrow$ check

$$\frac{K-K^{-1}}{q-q^{-1}}\cdot v_k = [n-2k]\, v_k$$

$\square$

<u>Construction of $R\in U\otimes U$</u>. (Drinfeld's idea)

$f(H)\,E = E\, f(H+2)$

$$U^{\geq 0} = \text{subalg gen by } \{H,E\}$$

$$U^{\leq 0} = \text{subalg gen by } \{H,F\} \leftarrow f(H)\,F = F\,f(H-2)$$

$\left(U^{\geq 0},\ U^{\leq 0}\ \text{are closed under } \Delta \right)$

There is a (unique) pairing

$$\mathcal{U}^{\leq 0} \times \mathcal{U}^{\geq 0} \longrightarrow \text{Scalars} \qquad \text{s.t.}$$

Hopf pairing

- $(1, x) = \varepsilon(x) \qquad \forall\, x \in \mathcal{U}^{\geq 0}$

  $(y, 1) = \varepsilon(y) \qquad \forall\, y \in \mathcal{U}^{\leq 0}$

- $(y, x_1 x_2) = (\Delta(y), x_1 \otimes x_2)$

  $(a_1 \otimes a_2, b_1 \otimes b_2) = (a_1, b_1)(a_2, b_2)$

- $(y_1 y_2, x) = (y_1 \otimes y_2, \Delta^{op}(x))$

- $(H, H) := \dfrac{2}{\ln(q)} = \dfrac{4}{\hbar}$

  $(F, E) := \dfrac{1}{q - q^{-1}} = \dfrac{1}{\hbar} \cdot (1 + O(\hbar))$

So Scalars $= \mathbb{C}((\hbar))$.

Drinfeld - $R \in \mathcal{U}^{\leq 0} \otimes \mathcal{U}^{\geq 0}$ is the canonical tensor of the pairing.

Meaning - if $\{A_k\}$ is a basis of $\mathcal{U}^{\leq 0}$

& $\{B_k\}$ = basis of $\mathcal{U}^{\geq 0}$ dual to $\{A_k\}$,

Then $R = \sum A_k \otimes B_k$

<u>Lemma</u>: This R satisfies cabling identities

$$\Delta \otimes \mathrm{id}(R) = R_{13} R_{23} \quad \text{in} \quad \mathcal{U}^{\leq 0} \otimes \mathcal{U}^{\leq 0} \otimes \mathcal{U}^{\geq 0}$$

$$\mathrm{id} \otimes \Delta (R) = R_{13} R_{12} \quad \text{in} \quad \mathcal{U}^{\leq 0} \otimes \mathcal{U}^{\geq 0} \otimes \mathcal{U}^{\geq 0}$$

pf Pair both sides w/ a typical elt

$$B_k \otimes B_\ell \otimes A_s \in \mathcal{U}^{\geq 0} \otimes \mathcal{U}^{\geq 0} \otimes \mathcal{U}^{\leq 0}$$

$$(\Delta \otimes \mathrm{id}(R),\ B_k \otimes B_\ell \otimes A_s)$$

$$= (\Delta(A_s),\ B_k \otimes B_\ell)$$

equal by axiom

$$(\underset{A_k \otimes B_k}{R_{13}}\ \underset{A_\ell \otimes B_\ell}{R_{23}},\ B_k \otimes B_\ell \otimes A_s) = (B_k \cdot B_\ell,\ A_s)$$

$\mathcal{U}^{\geq 0}$ - basis $\{H^a E^b : a, b \in \mathbb{Z}_{\geq 0}\}$

$\mathcal{U}^{\leq 0}$ - basis $\{H^c F^d : c, d \in \mathbb{Z}_{\geq 0}\}$

Compute $(H^c F^d, H^a E^b) = ?$

Step 1: $(H^n, H^m) = \delta_{mn} \cdot \dfrac{n!\, 2^n}{\ln(q)^n} = \dfrac{n!}{t^n}$ where $t = \dfrac{\ln(q)}{2}$.

pf $\overset{\shortparallel}{(\underbrace{H \otimes \cdots \otimes H}_{n},\ \Delta^{(n)}(H^m))}$ $\quad \Delta^{(n)} : \mathcal{U} \to \mathcal{U}^{\otimes n}$ via $\underbrace{\Delta^{(n+1)} = (\Delta \otimes \mathrm{id}) \circ \Delta^{(n)}}_{\text{co-assoc makes this canonical.}}$

Now $\Delta^{(n)}(H) = \displaystyle\sum_{j=0}^{n-1} 1^{\otimes j} \otimes H \otimes 1^{\otimes n-j-1}$

(j+1) st spot

$= \displaystyle\sum^{n} \ldots^{(j)}$ [assume $m < n$]

$$= \sum_{j=1}^{n} H^{(j)} \quad \text{(j+1) st spot} \qquad [\text{assume } m \le n].$$

$j=0$

$$(\Delta^{(n)}(H))^m = (H^{(1)} + H^{(2)} + \cdots + H^{(n)})^m$$

If $m<n$, then $(H^{(1)} + \cdots + H^{(n)})^m$ does not have $H^{(1)} H^{(2)} \cdots H^{(n)} = H \otimes \cdots \otimes H$

expanded involves monomials w/ 1 at some ~~tensor~~ component. so

$$(H \otimes \cdots \otimes H, (H^{(1)} + \cdots + H^{(n)})^m) = 0.$$

If $n=m$, $= n! \, (H \otimes \cdots \otimes H, H^{(1)} \cdots H^{(n)})$

$$= n! \, (H,H)^n$$

so $(H^n, H^m) = \delta_{n,m} \, n! \, (H,H)^n = \delta_{n,m} \left(\frac{2}{\ln(q)}\right)^n n!$.

Step 2 $(F^n, E^m) = \delta_{n,m} \dfrac{[n]! \, q^{-\frac{n(n-1)}{2}}}{(q-q^{-1})^n}$. $\qquad ([n]! = [n][n-1] \cdots [1])$.

· True by defn for $n=m=1$.

$$(F^n, E^n) = (F \cdot F^{n-1}, E^n) = (F \otimes F^{n-1}, \Delta^{op}(E^n))$$

$$(\Delta^{op}(E))^n = (E \otimes K + 1 \otimes E)^n$$

want terms w/ E on 1^{st} $\otimes$-component.

$\sum^{n-1}$ ... $n-1-i$ ... $n-1$

[illegible] w/ E on 1^{st} $\otimes$-component.

$$\sum_{i=0}^{n-1} E \otimes E^i K E^{n-1-i} = E \otimes K E^{n-1} \sum_{i=0}^{n-1} q^{-2i}$$

$EK = q^{-2} KE$

$$= \frac{1-q^{-2n}}{1-q^{-2}} \cdot E \otimes K E^{n-1}$$

$$= q^{-n+1} [n] \; E \otimes K E^{n-1}$$

$$(F^n, E^n) = [n] q^{-n+1} \cdot \frac{1}{q-q^{-1}} \underbrace{(F^{n-1}, K \cdot E^{n-1})}_{\text{// exercise}}$$

$$= \frac{[n] q^{-n+1}}{q-q^{-1}} (F^{n-1}, E^{n-1})$$

✓ by induction.

exercise: $(H^a F^b, H^c E^d) = (H^a, H^c) \cdot (F^b, E^d)$

$$= \delta_{ac} \delta_{bd} \frac{a! \, 2^a}{(\ln q)^a} \frac{[b]! \, q^{-\frac{b(b-1)}{2}}}{(q-q^{-1})^b}$$

$$R = \sum_{a,b \geq 0} \left(\frac{(\ln q)^a}{a! \, 2^a} H^a \otimes H^a \right) \cdot \left(\frac{(q-q^{-1})^b}{[b]!} q^{\frac{b(b-1)}{2}} F^b \otimes E^b \right)$$

$$= q^{\frac{H \otimes H}{2}} \cdot \left(\sum_{n \geq 0} q^{\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]!} (F \otimes E)^n \right)$$

$$\exp_q(x) \stackrel{\text{def}}{=} \sum_{n \geq 0} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]!}$$

so $$\boxed{\mathcal{R} = q^{\frac{H \otimes H}{2}} \cdot \exp_q\left((q-q^{-1}) F \otimes E\right)}$$