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Braided Tensor Categories

> Quantum gp $\mathcal{U}_h(\mathfrak{g})$ assoc to
>
> $\mathfrak{g}$ – simple lie alg (corr to root system R)
>
> are quasi-triangular Hopf algebras:

$(A, \Delta, \varepsilon, R)$ – q-t bialg.

- A: unital assoc alg $/\mathbb{C}$
- $\Delta : A \longrightarrow A \otimes A$ alg hom. s.t.

  $(\Delta \otimes id)(\Delta(x)) = (id \otimes \Delta)(\Delta(x)) \quad \forall\, x \in A.$

- $\varepsilon : A \longrightarrow \mathbb{C}$ alg hom.

  $(\varepsilon \otimes id)(\Delta(x)) = x = (id \otimes \varepsilon)(\Delta(x))$

- $R \in A \otimes A$ invertible s.t.

  (intertwining eqn) $\Delta^{op}(x) = R \cdot \Delta(x) \cdot R^{-1}$

(cabling identities) $(\Delta \otimes id)(R) = R_{13} \cdot R_{23}$

$(id \otimes \Delta)(R) = R_{13} \cdot R_{12}$

Bialgebra ⇝ Hopf algebra

Antipode $S : A \longrightarrow A$ alg anti-auto $(S(ab) = S(b)S(a))$ s.t.

$$\text{mult}((S \otimes id)(\Delta(x))) = \varepsilon(x) = \text{mult}((id \otimes S)(\Delta(x))) \quad \forall\ x \in A.$$

eg $\mathcal{U}(\mathfrak{g})$ $\quad$ ($\mathfrak{g}$ lie alg $/\mathbb{C}$)

↑ enveloping alg of $\mathfrak{g}$

$\Delta(x) = x \otimes 1 + 1 \otimes x$

$\varepsilon(x) = 0$

$S(x) = -x$

$\left[\ \mathfrak{g} \circlearrowright V \rightsquigarrow \mathfrak{g} \circlearrowright V^* \text{ by } (x \cdot \xi)(v) = -\xi(x \cdot v).\ \right]$

Remark: Antipode S allows us to define duals

$A \circlearrowright V \rightsquigarrow A \circlearrowright V^* \quad (a \cdot \xi)(v) = \xi(S(a) \cdot v)$

$$A \circlearrowright {}^{*}V \qquad (a\cdot\xi)(v) = \xi(S^{-1}(a)\cdot v)$$

Left/Right duals

<u>Lemma</u>: $\varepsilon\otimes id(R) = 1$

$S\otimes id(R) = R^{-1}$

pf

$$\begin{tikzcd}
\Delta\otimes 1(R) = R_{13}\cdot R_{23} \arrow[r, "S\otimes id\otimes id", "\text{multiply first 2}"'] \arrow[d, "\varepsilon\otimes id\otimes id"] & (\varepsilon\otimes id)(R) = (S\otimes id)(R)\cdot R. \quad \square \\
R = (\varepsilon\otimes id)(R)\cdot R & 1 \arrow[u, equal]
\end{tikzcd}$$

$$\Rightarrow \quad (\varepsilon\otimes id)(R) = 1 .$$

<u>Prop</u> R satisfies the following (in $A\otimes A\otimes A$)

$$R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12}$$

(Yang-Baxter eqn)

pf

$$\begin{gathered}
R_{12}\,\underbrace{R_{13}\,R_{23}}_{\shortparallel} \\
R_{12}\quad(\Delta\otimes id)(R) \\
\shortparallel \\
(\Delta^{op}\otimes id)(R)\cdot R_{12} \;=\; R_{23}\,R_{13}\,R_{23} \qquad \square
\end{gathered}$$

If $A \circlearrowright V$ and $n \in \mathbb{Z}_{\geqslant 2}$ then

$$B_n \longrightarrow GL(V^{\otimes n})$$

$$T_i \longmapsto (i\ \ i{+}1)\ R_{i,i+1}$$

is a gp hom.

pf $\quad T_i T_j = T_j T_i$ if $|i-j| \geqslant 2$ ✓

$$T_1 T_2 T_1 = T_2 T_1 T_2$$

$$(1\,2)\,R_{12}\ (2\,3)\,R_{23}\ (1\,2)\,R_{12} \underset{?}{=} (2\,3)\,R_{23}\ (1\,2)\,R_{12}\ (2\,3)\,R_{23}$$

Yes by moving perms & YB.

<u>Definition</u> $\mathcal{U}_\hbar(sl_2)$ quantum sl_2.

unital assoc alg over $\mathbb{C}[\![\hbar]\!]$ which is $\hbar$-adically complete

Generators $\quad H, E, F.$

Relations

$$[H,E] = 2E$$

$$[H,F] = -2F$$

$$[E,F] = \frac{e^{\frac{\hbar}{2}H} - e^{-\frac{\hbar}{2}H}}{e^{\frac{\hbar}{2}} - e^{-\frac{\hbar}{2}}} = H + O(\hbar^2)$$

$$q = e^{\hbar/2}, \quad K = e^{\frac{\hbar}{2}H} \quad \rightsquigarrow \quad [E,F] = \underline{K - K^{-1}}$$

$q - q^{-1}$.

$$\Big([H, E] = 2E$$

$$KEK^{-1} = \mathrm{Ad}\left(e^{\frac{\hbar}{2}H}\right) \cdot E$$

$$= \exp\left(\tfrac{\hbar}{2}\,\mathrm{ad}(H)\right) \cdot E = e^{\hbar} E = q^2 E \Big)$$

$$\mathrm{Ad}(e^{a}) \cdot b = \exp(\mathrm{ad}(a)) \cdot b$$

<u>Coproduct</u> $\Delta : U_\hbar(sl_2) \longrightarrow U_\hbar(sl_2)^{\otimes 2}$

$$\Delta(H) = H \otimes 1 + 1 \otimes H$$

$$\Delta(E) = E \otimes 1 + K \otimes E$$

$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$$

Thus Δ extends to an alg hom.

$$\left(KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F \right)$$

<u>Counit</u>: $\varepsilon(H) = \varepsilon(E) = \varepsilon(F) = 0$

<u>Antipode</u>: $S(H) = -H$

$$S(E) = -K^{-1}E$$

$$S(F) = -FK$$

extends to alg antihom.

(Satisfies

$$\mathrm{mult}(S \otimes \mathrm{id}(\Delta(x))) = \varepsilon(x)$$

$$= \mathrm{mult}(\mathrm{id} \otimes S(\Delta(x)))$$

Need $R \in U_\hbar(sl_2)^{\otimes 2}$ satisfying intertwining eq^n & cabling id.

Then $(U_\hbar(sl_2), \Delta, \varepsilon, R; S) = $ q-Δ-Hopf alg