10/21

Monday, October 21, 2019 13:48

Braided Tensor categories from KZ equation

$\mathfrak{g}$: finite dim lie alg $/\mathbb{C}$

$(\cdot,\cdot) : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{C}$ nondegen, symmetric, bilinear, invariant form

$\Omega \in \mathfrak{g}\otimes\mathfrak{g}$ casimir tensor.

$$\nabla KZ_n = d - \kappa \sum_{1\le i<j\le n} \frac{d(z_i - z_j)}{z_i - z_j}\Omega_{ij} \quad , \qquad \kappa \text{ arbitrary parameter } \in \mathbb{C}$$

(for a function $F(z_1,\ldots,z_n) \in V_1\otimes\cdots\otimes V_n$, $V_i \in Rep_{fd}(\mathfrak{g})$)

$$[x\otimes 1 + 1\otimes x, \Omega] = 0 \quad \forall x, \qquad \Omega_{21} = \Omega$$

$$\mathcal{U}(\mathfrak{g}) = \frac{\text{free assoc alg over } \mathfrak{g}}{\text{2-sided ideal containing } x\cdot y - y\cdot x - [x,y] \quad \forall x,y\in\mathfrak{g}}$$

Alg hom-s $\Delta : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$

$$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \forall x \in \mathfrak{g}.$$

$$\varepsilon : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}$$

$$\varepsilon(x) = 0 \qquad \forall x \in \mathfrak{g} \qquad \varepsilon(1) = 1$$

Drinfeld Approach

$$\left(\mathcal{U}(\mathfrak{g})[[\hbar]], \Delta, \varepsilon, R_{KZ}, \Phi_{KZ}\right)$$

is a q-t-q-b

Notation $\quad k = \frac{\hbar}{2\pi i}$, $\quad \hbar$ - formal parameter

$$R_{KZ} = e^{\frac{\hbar}{2}\Omega} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}[[\hbar]]$$

Thm

$$\left(\mathrm{Rep}_{fd}(\mathfrak{g}), \otimes, \underset{(1\,2)\circ R_{KZ}}{\overset{c}{\|}}, \underset{\Phi_{KZ}}{\overset{a}{\|}}\right)$$ is a braided tensor category

$(k \in \mathbb{C} \setminus \mathbb{Q}$ "generic"$) \longrightarrow$ Kazhdan Lusztig

Pf $\quad$ recall Φ_{KZ} = associator for $F'(z) = k\left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right)F$.

$$\varepsilon \otimes 1(\Delta(a)) = a = 1 \otimes \varepsilon(\Delta(a)) \qquad \forall a \in \mathcal{U}(\mathfrak{g})$$

$1 \otimes \varepsilon \otimes 1 \, (\Phi) = 1 \otimes 1$ $\qquad$ $1 \otimes \varepsilon \otimes 1 : \begin{cases} \Omega_{12} \longmapsto 0 \\ \Omega_{23} \longmapsto 0 \end{cases}$

$\Phi(A=0, B=0) = 1$

- $\varepsilon \otimes 1 (R_{KZ}) = 1 = 1 \otimes \varepsilon (R_{KZ})$

- $\Delta^{op}(x) = R \, \Delta(x) \, R^{-1} \quad \forall x \in \mathcal{U}(\mathfrak{g})$

- $1 \otimes \Delta(\Delta(x)) = \Phi \; \Delta \otimes 1 (\Delta(x)) \; \Phi^{-1}$

<u>Left</u> - $\boxed{\text{pentagon axiom}}$; Hexagon axioms

$$(1 \otimes 1 \otimes \Delta)(\Phi) \cdot (\Delta \otimes 1 \otimes 1)(\Phi)$$
$$= (1 \otimes \Phi) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (\Phi \otimes 1)$$

<u>Idea:</u> For every bracketing b on 4 letters there is a canonical soln Ψ_b of $\nabla_{KZ_4} \Psi_b = 0$ s.t. $\Phi_{b',b} = \Psi_{b'}^{-1} \cdot \Psi_b$.

eg $n=3$

Asymptotic Zone

$(\cdot\,\cdot)\cdot$	$\|z_2-z_1\| \ll \|z_3-z_1\|$	ψ_1
$\cdot(\cdot\,\cdot)$	$\|z_3-z_2\| \ll \|z_3-z_1\|$	ψ_2

$$\psi_1 = H\left(\frac{z_2-z_1}{z_3-z_1}\right)\left(\frac{z_2-z_1}{z_3-z_1}\right)^{k\Omega_{12}}(z_3-z_1)^{k(\Omega_{12}+\Omega_{23}+\Omega_{13})}$$

$$\psi_2 = \tilde{H}\left(\frac{z_3-z_2}{z_3-z_1}\right)\left(\frac{z_3-z_2}{z_3-z_1}\right)^{k\Omega_{23}}(z_3-z_1)^{k(\Omega_{12}+\Omega_{23}+\Omega_{13})}$$

$$\psi_1 = \psi_2\,\Phi_{KZ}$$

$n=4$

$b_1 = ((\cdot\,\cdot)\cdot)\cdot$

$$\underbrace{z_2-z_1}_{uvw} \ll \underbrace{z_3-z_1}_{uv} \ll \underbrace{z_4-z_1}_{u}$$

rewrite ∇_{KZ_4} in these variables

$$\psi_1 = \text{Holomorphic}\cdot u^{k\Omega_{[14]}}\cdot v^{k\Omega_{[13]}}\cdot w^{k\Omega_{12}}$$

get a table of asymptotics for solution.