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sl_2 – representation theory

$sl_2 = \mathbb{C}$-span of h, e, f

$[h, e] = 2e$ $\quad [e, f] = h.$

$[h, f] = -2f$

L_n : $(n-1)$-dim'l $\mathbb{C}$-v.s. w/ basis $\{v_0, \dots, v_n\}$;

sl_2-action given by

$$h \cdot v_j = (n-2j) v_j$$

$$e v_j = (n-j+1) v_{j-1}$$

$$f v_j = (j+1) v_{j+1}$$

Last time $\quad sl_2 \circlearrowright V \overset{\text{f.d.}}{\text{irr. repn}} \Rightarrow V \cong L_n \quad (n = \dim V - 1).$

$sl_2 \circlearrowright V$ f.d.

Let $v \in V \setminus \{0\}$ s.t. $hv = \mu v$ $(\mu \in \mathbb{C})$.

Then $\mu \in \mathbb{Z}_{\geq 0}$, and $f^{\mu+1} v = 0$.

(*) $\quad ef^k = f^k e + k f^{k-1}(h-k+1) \quad (ef - fe = h)$

(*) : $[x,-]$ obeys Leibniz rule

$$[e, f^k] = \sum_{j=0}^{k-1} f^{k-1-j} \underbrace{[e,f]}_{h} f^j$$

$$[h, f^j] = -2j f^j$$

$$= \sum_{j=0}^{k-1} f^{k-1-j} (f^j h - 2j f^j)$$

$$= k \cdot f^{k-1} h - 2 \left(\sum_{j=0}^{k-1} j \right) f^{k-1}$$

$$\frac{k(k-1)}{2}$$

$$= k f^{k-1} (h - k + 1)$$

~~pf~~ let $m \in \mathbb{Z}_{\geq 0}$ s.t $f^m v \neq 0$, $f^{m+1} v = 0$.

$$0 = e \cdot f^{m+1} v = \cancel{f^{m+1} e v}_{0} + (m+1) f^m (h-m) v \Rightarrow \mu - m = 0$$

$$\Rightarrow m = \mu.$$

<u>Remark</u> $L_n^* \cong L_n \quad \forall n \in \mathbb{Z}_{\geq 0}$.

<u>Theorem</u> Any f.d. repn of sl_2 = direct sum of irreducibles.

Casimir element

$$C = \frac{h^2}{2} + ef + fe$$

(more precisely, $sl_2 \circlearrowright V$ f.d.

then $C_V \in \operatorname{End} V$)

<u>Lemma</u> $\forall x \in sl_2, \quad [x, C] = 0$

~~pf~~ $[h, C] = \underbrace{[h, \frac{h^2}{2}]}_{0} + [h,e]f + e[h,f]$

$+ [h,f]e + f[h,e]$

$= 2ef - 2ef - 2fe + 2fe = 0$

$[e, C] = \frac{1}{2}[e, h^2] + e[e,f] + [e,f]e$

$= \frac{1}{2}\Big(\underbrace{[e,h]h}_{-2eh} + \underbrace{h[e,h]}_{-2he}\Big) + eh + he$

similarly for f.

<u>Cor</u> for $V \in V$ irred. f.d. repn, $C_V = \lambda_V \mathrm{Id}_V$

(& Schur's Lemma)

$$\left.\begin{matrix}\lambda_V \\ V=L_n\end{matrix}\right] \quad C = \left(\frac{h^2}{2} + 2fe + h\right) v_0$$

$$= \left(\frac{n^2}{2} + n\right) v_0$$

So C_{L_n} acts by $\frac{n(n+2)}{2}$.

$$n(n+2) = m(m+2) \Rightarrow n=m \quad \text{or} \quad \underbrace{n+m=-2}_{\text{can't happen, } n,m\geq 0.}$$

Prop Every short exact sequence of f.d. sl_2-repns splits:

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0 \qquad \text{seq. of } sl_2\text{- intertwiners}$$

$$\Rightarrow \quad V \cong V' \oplus V'' \qquad \text{as } sl_2\text{- repns.}$$

Prop $\Rightarrow$ Thm by inducting on dimension.

Assume $V' = L_n$, $V'' = L_m$.

Case 1 $m \neq n$, assume $n < m$ (if not, take dual).

$$0 \longrightarrow L_n \longrightarrow V \longrightarrow L_m \longrightarrow 0$$

$$\overline{V} = \{ v \in V \mid C \cdot v = \frac{m(m+2)}{2} v \}$$

$$\begin{array}{l} \cap \quad \longleftarrow \text{subrepn,} \\ V \end{array}$$

$$L_n \cap \overline{V} = (0) \quad \text{because} \quad C|_{L_n} = \frac{n(n+2)}{2} \neq \frac{m(m+2)}{2}$$

so $\overline{V} \cong L_m$

Case 2 $m = n$

$$0 \longrightarrow \overset{(1)}{L_n} \xrightarrow{\ i\ } V \xrightarrow{\ \pi\ } \overset{(2)}{L_n} \longrightarrow 0$$

$$\begin{array}{cccccc}
 & v_0^1 & \longmapsto & v_0 & \underset{w_0}{\text{choose}} \longmapsto & v_0^2 \\
f \downarrow & v_1^1 & \longmapsto & v_1 & & v_1^2 \\
f \downarrow & \vdots & & & & \vdots \\
f \downarrow & v_n^1 & \longmapsto & v_n & & v_n^2
\end{array}$$

Let

$$w_\ell = \frac{f}{\ell!} w_0$$

What can go wrong? $\quad h v_j = (n-2j) v_j$

$$h w_0 = n w_0 + \lambda v_0$$

what if $\lambda \neq 0$?

Easy Check: we have

$$e \cdot w_j = (n-j+1) w_{j-1} + \lambda v_{j-1}$$

$$f \cdot w_j = (j+1)\, w_{j+1}$$

same λ

$$h \cdot w_j = (n-2j)\, w_j + \lambda v_j$$

$\underline{\lambda = 0:}$ $f^{n+1} w_0 = 0,$

$$0 = e f^{n+1} w_0 = \cancel{f^{n+1} e w_0} + (n+1) f^n (h-n) w_0$$

$$0 = f^n (\cancel{n w_0} + \lambda v_0 - \cancel{n w_0})$$

$$= \lambda \cdot (f^n v_0) \neq 0$$

$$\Rightarrow \lambda = 0.$$

(Thm: Weyl's complete reducibility thm).

$\underline{\text{Cor}}$ $\mathfrak{sl}_2 \circlearrowright V$ f.d. $\Rightarrow$ $h \in \mathrm{End}(V)$ is $\underline{\text{semisimple}}$.

and its eigenvalues are integers.

V has a basis of eigenvectors of h

$\underline{\text{Notation}}$ $V[k]$ = subspace w/ h-eigenvalue k

$$= \{ v \in V \mid hv = kv \}$$

weight space of weight k.

$\underline{\text{not}}$ subrepns.

$$V = \bigoplus_{k \in \mathbb{Z}} V[k]$$

$\dim V[k]$ = multiplicity of weight k.

$$\underbrace{P(V)}_{\text{weights of } V.} = \{k \in \mathbb{Z} \mid V[k] \neq 0\} \subset \mathbb{Z}$$

$$P(L_n) = \{-n, -n+2, \dots, n-2, n\} \subset \mathbb{Z}.$$

<u>Remark</u> Thm is false for inf. dim repns.

Let $\lambda \in \mathbb{C}$, M_λ is defined as a vecter space

$= \mathbb{C}\text{-span}\{m_0, m_1, \dots\}$

$\mathfrak{sl}_2 \circlearrowright M_\lambda$:

$$h \cdot m_k = (\lambda - 2k) m_k$$
$$e \cdot m_k = (\lambda - k + 1) m_{k-1}$$
$$f \cdot m_k = (k+1) m_{k+1}$$

<u>Ex</u> if $\lambda \notin \mathbb{Z}_{\geq 0}$, then M_λ is irred.

if $\lambda \in \mathbb{Z}_{\geq 0}$, we get a non-split s.e.s.

$$0 \longrightarrow M_{-\lambda-2} \longrightarrow M_\lambda \longrightarrow L_\lambda \longrightarrow 0$$

Example of Tensor Product

basis $\uparrow, \downarrow$

$$L_n \otimes L_1 = L_{n+1} \oplus L_{n-1} \qquad (L_1 \cong \mathbb{C}^2 \circlearrowleft sl_2)$$

$$\forall n \geq 1, \qquad L_0 \otimes V = V \quad \text{since } L_0 \cong \mathbb{C} \text{ is } 1\text{-dim'l.}$$

trivial repn

	basis of $L_n \otimes L_1$			weight
generates L_{n+1}	$v_0 \otimes \uparrow$			$n+1$
	$v_1 \otimes \uparrow$	$v_0 \otimes \downarrow$	one left over	$n-1$
	$\vdots$	$\vdots$	L_{n-1}	$\vdots$
	$v_n \otimes \uparrow$	$v_{n-1} \otimes \downarrow$		$-n+1$
		$v_n \otimes \downarrow$		$-n-1$

(weights = eigenvalues of h)

Facts (1) $sl_2 \circlearrowright V$ f.d.

$$\#\text{ irreducible summands} = \dim(\mathrm{Ker}(e))$$

$$V^\circ = \{v \in V \mid ev = 0\}$$

(2) $P(V) = -P(V)$

(3) $\dim V[k] = \dim V[-k]$

(4) there is an element $s \in GL(V)$

s.t. $s : V[k] \xrightarrow{\sim} V[-k] \quad \forall\, k \in \mathbb{Z}$

$$s \text{ "="} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2(\mathbb{C})$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$s = \exp(e)\, \exp(-f)\, \exp(e) \in GL(V)$$