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Friday, October 18, 2019 13:51

Braided Tensor Category $(\mathcal{C}, \otimes, c, a)$

- $\mathcal{C}$ is a $\mathbb{C}$-linear category
- $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a bifunctor
- a: assoc constraint satisfying pentagon axiom
- c: commutativity constraint satisfying hexagon axioms

<u>Unit Object</u> $(\mathbb{1},\ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X,\ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X)$

s.t.

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{Id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{Id}_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes \mathbb{1} \arrow[rr, "c_{X,\mathbb{1}}"] \arrow[dr, "r_X"'] & & \mathbb{1} \otimes X \arrow[dl, "l_X"] \\
& X &
\end{tikzcd}$$

A quasibialgebra $(A, \Delta, \varepsilon, \Phi)$ consists of

(i) unital assoc alg $A / \mathbb{C}$.

(ii) $\Delta : A \longrightarrow A \otimes A$

(iii) $\varepsilon : A \longrightarrow \mathbb{C}$

(iv) $\Phi \in A^{\otimes 3}$ invertible

s.t.

(1) $(1 \otimes \Delta)(\Delta(a)) = \Phi \cdot (\Delta \otimes 1 \ (\Delta(a))) \cdot \Phi^{-1}$

(2) $\varepsilon \otimes 1 \ (\Delta(a)) = a = 1 \otimes \varepsilon \ (\Delta(a))$

(3) $1 \otimes \varepsilon \otimes 1 \ (\Phi) = 1 \otimes 1$

(4) Pentagon Axiom

$$\underbrace{(1 \otimes 1 \otimes \Delta)(\Phi)} \cdot \underbrace{(\Delta \otimes 1 \otimes 1)(\Phi)}$$
$$= \underbrace{(1 \otimes \Phi)} \cdot \underbrace{(1 \otimes \Delta \otimes 1)(\Phi)} \cdot \underbrace{(\Phi \otimes 1)} \quad \text{in } A^{\otimes 4}$$

componentwise multiplication

A : quasibialgebra $\rightsquigarrow$ $\mathrm{Rep}(A)$ is a tensor category.

For $\mathrm{Rep}(A)$ to be a Braided tensor category,

we need $R \in A \otimes A$ invertible

· We want $\quad C_{XY} = (1\,2) \circ R_{X,Y}$

<u>Definition</u> A quasi-triangular quasi-bialgebra $(A, \Delta, \varepsilon, R, \Phi)$ is the data of

(i) Quasi-bialgebra

(ii) $R \in A \otimes A$ invertible (called R-matrix)

s.t.

(1) $\Delta^{op}(a) = R \cdot \Delta(a) \cdot R^{-1}$

$$V \otimes W \xrightarrow{(1\,2) \circ R_{V,W}} W \otimes V$$

is A-linear

$$\begin{tikzcd} R \in A \otimes A \arrow[r] \arrow[rr, bend left, mapsto] & \mathrm{End}(V \otimes W) & R_{V,W} \end{tikzcd}$$

So

$$(1\,2) \cdot R \cdot \Delta(a) = \Delta(a)\,(12)\,R$$

$$R\,\Delta(a)\,R^{-1} = \Delta^{op}(a)$$

(2) Hexagon Axiom / Cabling identities

in $A^{\otimes 3}$

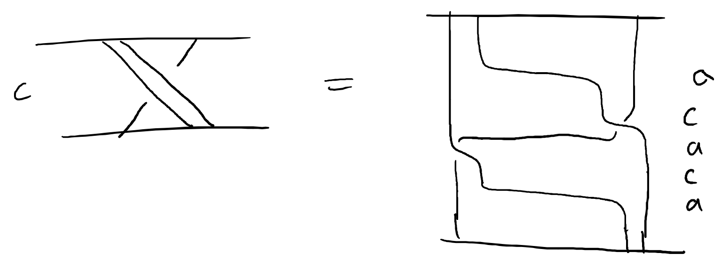
$$\Delta \otimes 1\,(R) = \Phi_{312}\ R_{13}\ \Phi_{132}^{-1}\ R_{23}\ \Phi_{123}$$

$$\Delta\otimes 1\,(R) = \Phi_{312}\; R_{13}\; \Phi_{132}\; \underbrace{R_{23}}_{=1\otimes R}\; \Phi_{123}$$

$$\Phi = \sum_k a_k\otimes b_k\otimes c_k \;\rightsquigarrow\; \Phi_{312} = \sum b_k\otimes c_k\otimes a_k$$

$$\Phi = \Phi_{123} \qquad\qquad \Phi_{132} = \sum a_k\otimes c_k\otimes b_k$$

$$R = \sum \alpha_k\otimes\beta_k \;\rightsquigarrow\; R_{13} = \sum \alpha_k\otimes 1\otimes\beta_k$$



$$\cancel{(1\;2\;3)}\;\Delta\otimes 1\,(R) = \Phi_{123}\underbrace{(1\;2)}\; R_{12}\; \Phi^{-1}_{123}\,(2\;3)\; R_{23}\,\Phi_{123}$$

$$= (1\;2)\;\Phi_{213}\; R_{12}\,\Phi^{-1}_{123}\;\underbrace{(2\;3)}\; R_{23}\;\Phi_{123}$$

$$= \cancel{(1\;2)(2\;3)}\;\Phi_{312}R_{13}\;\Phi^{-1}_{132}\;R_{23}\;\Phi_{123}\,.$$

$$1\otimes\Delta\,(R) = \Phi^{-1}_{231}\;R_{13}\,\Phi_{213}\;R_{12}\;\Phi^{-1}_{123}$$

(3) $\varepsilon \otimes 1\,(R) = 1 \otimes 1 = 1 \otimes \varepsilon\,(R)$.

If $R^{-1} = R_{21}$, then triangular.

$(A, \Delta, \varepsilon, R, \Phi)$ $\rightsquigarrow$ Rep A is a braided tensor category
q-t-q-b

bialgebra $\rightsquigarrow$ Hopf algebra needs duals!

Let $\mathcal{C}$, $\mathcal{D}$ be two tensor categories

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an additive functor.

A <u>tensor structure</u> on F, denoted by J,

is natural iso-s

$$F(X \otimes Y) \xrightarrow{J_{X,Y}} F(X) \otimes F(Y)$$

$\forall\, X, Y \in \mathcal{C}$, natural in X & Y.

$$
\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow J"] & \\
\mathcal{C} \times \mathcal{C} \arrow[ur, "\otimes"] \arrow[r, "F \times F"'] & \mathcal{D} \times \mathcal{D} \arrow[r, "\otimes"'] & \mathcal{D}
\end{tikzcd}
$$

s.t. the following diagram commutes (twist / cocycle eqn)

$$
\begin{tikzcd}
F((X \otimes Y) \otimes Z) \arrow[r, "F(a^{\mathcal{C}}_{X,Y,Z})"'] \arrow[d, "J_{X \otimes Y, Z}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "J"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "J \otimes \mathrm{Id}"] \arrow[r, phantom, "\circlearrowright"] & F(X) \otimes F(Y \otimes Z) \arrow[d, "\mathrm{Id} \otimes J"] \\
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "a^{\mathcal{D}}_{F(X), F(Y), F(Z)}"'] & F(X) \otimes (F(Y) \otimes F(Z))
\end{tikzcd}
$$

If $\mathcal{C}, \mathcal{D}$ are also braided, we say J is compatible w/ braidings if

$$F(X \otimes Y) \xrightarrow{F(c^{\mathcal{C}}_{X,Y})} F(Y \otimes X)$$

$$\begin{tikzcd} F(X \otimes Y) \arrow[r] \arrow[d, "J_{XY}"] \arrow[dr, phantom, "\circlearrowright"] & F(Y \otimes X) \arrow[d, "J_{YX}"] \\ F(X) \otimes F(Y) \arrow[r, "c^{\mathcal{D}}_{F(X), F(Y)}"'] & F(Y) \otimes F(X) \end{tikzcd}$$

Twisting a Quasi-bialg

$(A, \Delta, \varepsilon, \Phi)$ $\quad J \in A \otimes A$ invertible

$\rightsquigarrow$ a new quasi-bialgebra $(A, \Delta_J, \varepsilon, \Phi_J)$

$\forall x \in A \quad \Delta_J(x) := J \cdot \Delta(x) \cdot J^{-1}$

Axiom $\quad \varepsilon \otimes 1 (J) = 1 = 1 \otimes \varepsilon (J)$

$$\boxed{\Phi_J = (1 \otimes J) \cdot (1 \otimes \Delta(J)) \cdot \Phi \cdot (\Delta \otimes 1 (J))^{-1} (J \otimes 1)^{-1}}$$

Ex $\quad (A, \Delta_J, \varepsilon, \Phi_J)$ is a quasi-bialg

"Trivializing an Associator"

Given Φ, find J s.t. $\Phi_J = 1^{\otimes 3}$.

$$(A, \Delta, \varepsilon, R, \Phi) \rightsquigarrow (A, \Delta_J, \varepsilon, \underset{\underset{\textstyle J_{21} R J^{-1}}{\|}}{R_J}, \Phi_J$$

$\boxed{\text{twist of a q-t-q-b by } J}$

$$(\mathrm{Rep}\, A, \otimes, a) \xrightarrow{\mathrm{Id}} (\mathrm{Rep}\, A, \otimes_J, a_J)$$

J is a tensor structure on the identity functor.

<u>Ex</u> $(\mathrm{Rep}\, \mathfrak{g}, \otimes, c = (12), a = 1)$

$\mathfrak{g} \circlearrowright V \otimes W$ by $x \otimes 1 + 1 \otimes x$

$$\begin{aligned} V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longmapsto w \otimes v \end{aligned} \quad \text{is } \mathfrak{g}\text{-intertwiner}$$

$$(V_1 \otimes V_2) \otimes V_3 \xrightarrow{\sim} V_1 \otimes (V_2 \otimes V_3)$$

natural id
of vector sp

<u>Ex</u> $\mathfrak{g}$: Lie algebra / $\mathbb{C}$

$\mathcal{U}(\mathfrak{g})$ - unital assoc alg / $\mathbb{C}$

$$\mathrm{Rep}(\mathfrak{g}) = \mathrm{Rep}(\mathcal{U}(\mathfrak{g}))$$

(<u>eg</u> $\mathcal{U}(sl_2) = \dfrac{\mathbb{C}\langle e, f, h\rangle}{\text{2-sided ideal gen by } \begin{array}{l} he - eh - 2e \\ hf - fh - (-2f) \\ ef - fe - h \end{array}}$)

$h, e, f,$
$[h,e] = 2e$
$[h,f] = -2f$
$[e,f] = h$

$$\mathcal{U}(\mathfrak{g}) = \frac{\text{free assoc. algebra gen by } \mathfrak{g}}{x \cdot y - y \cdot x - [x,y]}$$

$\mathcal{U}(\mathfrak{g})$ is a q-t-q-b / co-comm bialg

$$\begin{cases} \Delta(x) = x \otimes 1 + 1 \otimes x \quad \forall\ x \in \mathfrak{g} \\ \varepsilon(x) = 0 \\ \Phi = 1^{\otimes 3} \end{cases}$$

$R = 1^{\otimes 2}$

$\cdot\ I\ ^{-}\ 1 \qquad R = 1^{\otimes 2}$

$\mathfrak{g}$ = quadratic $(\cdot,\cdot)$ $\quad \Omega \in \mathfrak{g}\otimes\mathfrak{g}$

Then

$$\left(\mathcal{U}(\mathfrak{g}),\ \Delta,\ \varepsilon,\ R_{KZ},\ \Phi_{KZ}\right) \text{ is also a q-t-q-b.}$$

$$R_{KZ} = e^{\pi i \hbar \Omega}$$

Thm K-D

$$\left(\mathcal{U}_\hbar(\mathfrak{g}),\ \Delta_\hbar,\ \varepsilon,\ R_\hbar,\ 1^{\otimes 3}\right) \text{ is another example.}$$

$\uparrow$

quantum groups