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Braided Tensor Categories

$\mathcal{C}$: category, $\mathbb{C}$-linear ; abelian

Hom$_{\mathcal{C}}(X,Y)$ is a f.d. $\mathbb{C}$-v.s.

and composition is $\mathbb{C}$-bilinear

- direct sums of objects exist
- Kernel / Cokernel exist
- Every bijection is iso

eg $\mathcal{C} = \mathrm{Rep}_{fd}(A)$ (A : unital assoc. algebra / $\mathbb{C}$)

Objects: V : f.d. vs / $\mathbb{C}$

$+ \ \pi : A \to \mathrm{End}_{\mathbb{C}}(V)$ alg hom

Morphisms:

$$\mathrm{Hom}_{\mathcal{C}}(V,W) = \left\{ \underset{\mathbb{C}\text{-linear}}{f : V \longrightarrow W} \;\middle|\; f(a\cdot v) = a \cdot f(v) \right\}$$

Tensor Category $\mathcal{C}$ together with

(1) $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ bifunctor (additive)

$(X,Y) \longmapsto X \otimes Y$

(2) Unit object $1_{\mathcal{C}} \in \mathcal{C}$

(3) Associativity Constraint a

$\forall X, Y, Z \in \mathcal{C}$, $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ natural in X, Y, Z

(i.e.

$$
\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "{\otimes, id}"] \arrow[d, "{id, \otimes}"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \arrow[dl, phantom, "\swarrow a"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}
$$

a is a natural trans of functors

$$\otimes \cdot (\otimes, id) \Rightarrow \otimes (id, \otimes) \quad)$$

* left & right unit constraints

$$\forall X \in \mathcal{C} \quad \begin{array}{l} l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \\ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \end{array} \quad \text{natural in } X.$$

<u>Axioms</u> <u>Pentagon Axiom</u>: $\forall X_1, X_2, X_3, X_4 \in \mathcal{C}$,

$$
\begin{tikzcd}
 & (X_1 \otimes X_2) \otimes (X_3 \otimes X_4) \arrow[dr, "{a_{X_1, X_2, X_3 \otimes X_4}}"] & \\
((X_1 \otimes X_2) \otimes X_3) \otimes X_4 \arrow[ur, "{a_{X_1 \otimes X_2, X_3, X_4}}"] \arrow[d, "{a_{X_1, X_2, X_3} \otimes id_{X_4}}"'] & \text{Commutes} & X_1 \otimes (X_2 \otimes (X_3 \otimes X_4)) \\
{} & & {} \arrow[u]
\end{tikzcd}
$$

Commutes

$$\downarrow a_{X_1,X_2,X_3} \otimes \mathrm{Id}_{X_4} \qquad\qquad \nearrow \mathrm{Id}_{X_1} \otimes a_{X_2,X_3,X_4}$$

$$(X_1 \otimes (X_2 \otimes X_3)) \otimes X_4 \xrightarrow[a_{X_1, X_2 \otimes X_3, X_4}]{} X_1 \otimes ((X_2 \otimes X_3) \otimes X_4)$$

a vs unit

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{Id}_Y"'] & \arrow[d, phantom, "\text{commutes}"] & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{Id}_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

When $\mathcal{C} = \mathrm{Rep}_{fd}(A)$

- Need an algebra hom $\Delta : A \longrightarrow A \otimes A$ (coproduct) (defines $\otimes$ on $\mathcal{C}$)

- Need an algebra hom $\varepsilon : A \longrightarrow \mathbb{C}$ (counit) (defines $\mathbb{1}_{\mathcal{C}}$)

- Need $\Phi \in A \otimes A \otimes A$ invertible (associator) (defines assoc. constraint)

These give:

$A \circlearrowright V_1, V_2 \qquad \pi_j : A \longrightarrow End(V_j)$

$$A \circlearrowright V_1 \otimes V_2 \quad \text{by} \quad \begin{array}{ccc} A & \longrightarrow & End(V_1 \otimes V_2) \\ \rotatebox{90}{$\in$} & & \\ a & \longmapsto & \pi_1 \otimes \pi_2 (\Delta(a)) \end{array}$$

$$\mathbb{1}_{\mathcal{C}} = \mathbb{C} \circlearrowleft A \quad \text{Via} \quad \varepsilon : A \longrightarrow \mathbb{C} = End_{\mathbb{C}}(\mathbb{C})$$

$$A \circlearrowright V_1, V_2, V_3 \quad \text{via} \quad A \xrightarrow{\pi_i} End(V_i)$$

$$a_{V_1, V_2, V_3} = \pi_1 \otimes \pi_2 \otimes \pi_3 (\Phi)$$

Axioms

Pentagon Axiom

$$(1 \otimes 1 \otimes \Delta)(\Phi) \cdot (\Delta \otimes 1 \otimes 1)(\Phi)$$

$$= (1 \otimes \Phi) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (\Phi \otimes 1)$$

Left/right unit

$$(\varepsilon \otimes 1) \circ \Delta(a) = a = (1 \otimes \varepsilon) \circ \Delta(a) \qquad \forall a \in A.$$

(Φ v.s. Δ): $(1 \otimes \Delta)(\Delta(a)) = \Phi (\Delta \otimes 1)(\Delta(a)) \Phi^{-1} \qquad \forall a \in A$

(assoc. vs unit): $(1 \otimes \varepsilon \otimes 1)(\Phi) = 1 \otimes 1$.

A unital assoc. algebra over $\mathbb{C}$

$(A, \Delta, \varepsilon, \Phi)$ is a quasi-bialgebra.

(if $\Phi = 1 \otimes 1 \otimes 1$, just bialgebra).

Back to general tensor categories

Mac Lane's Coherence Theorem

for $n \in \mathbb{Z}_{\geq 2}$, $X_1, \dots, X_n \in \mathcal{C}$ and $\underline{b}$ a complete bracketing on n letters,

we get $(X_1 \otimes \cdots \otimes X_n)_{\underline{b}} \in \mathcal{C}$.

Let $\mathcal{B}_n$ = set of all such $\underline{b}$.

Thm $\forall\, \underline{b}, \underline{b}' \in \mathcal{B}_n$, there is a unique extension of the assoc. constraint to an iso

$$(X_1 \otimes \cdots \otimes X_n)_{\underline{b}} \xrightarrow{a_{\underline{b}', \underline{b}}} (X_1 \otimes \cdots \otimes X_n)_{\underline{b}'}$$

$\underline{n=2}$ id

$\underline{n=3}$ original associator a or its inverse.

$\underline{n=4}$ There are 2 natural exts. but pentagon axiom says they're equal.

<u>Idea</u> construct a space $\mathcal{A}_n$ (Associahedron)

vertices (0-cells) = elements of B_n

edges (1-cells) = associativity

faces (2-cells) = Pentagons

$\mathcal{A}_n$ is connected $\Rightarrow$ existence.

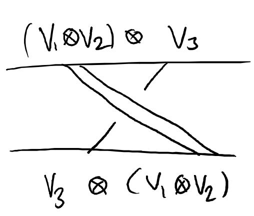
$\mathcal{A}_n$ is simply connected $\Rightarrow$ uniqueness.

<u>Braided Tensor Category</u>: $(\mathcal{C}, \otimes, a)$

A <u>commutativity constraint</u> on tensor category $\mathcal{C}$

is iso $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ (natural in X & Y).

Satisfying hexagon axioms



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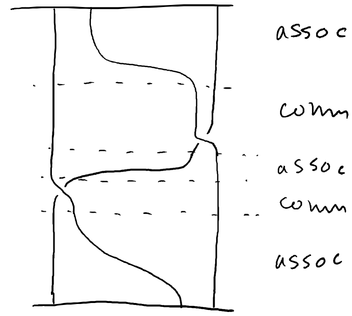


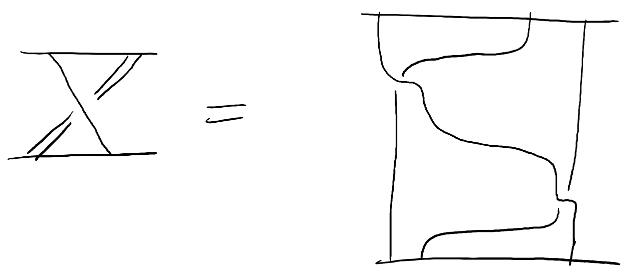
$c_{V_1 \otimes V_2, V_3}$

$c_{V_1 \otimes V_2, V_3}$

$\forall\ V_1, V_2, V_3 \in \mathcal{C}$,

$$\begin{tikzcd}
(V_1 \otimes V_2) \otimes V_3 \arrow[rr, "c_{V_1 \otimes V_2, V_3}"] \arrow[d, "a_{V_1, V_2, V_3}"] & & V_3 \otimes (V_1 \otimes V_2) \\
V_1 \otimes (V_2 \otimes V_3) \arrow[d, "\mathrm{Id}_{V_1} \otimes c_{V_2 V_3}"] & \text{commutes} & (V_3 \otimes V_1) \otimes V_2 \arrow[u, "a_{V_3 V_1 V_2}"'] \\
V_1 \otimes (V_3 \otimes V_2) \arrow[rr, "a^{-1}_{V_1 V_3 V_2}"'] & & (V_1 \otimes V_3) \otimes V_2 \arrow[u, "c_{V_1 V_3} \otimes \mathrm{Id}_{V_2}"']
\end{tikzcd}$$

And Hexagon$_2$:



If $(\mathcal{C}, \otimes, c, a)$ is a braided tensor category

and $n \in \mathbb{Z}_{\geq 2}$ and $V \in \mathcal{C}$,

then we automatically have an action of Artin's

braid gp B_n on $V_{\underline{b}}^{\otimes n}$ $(\forall \underline{b} \in B_n)$ as follows braid gp $\underset{\underset{\langle T_1, \dots, T_{n-1} \mid \text{rel}^{n}\text{'s} \rangle}{\|}}{B_n}$ on $V_{\underline{b}}^{\otimes n}$ $(\forall \underline{b} \in \mathcal{B}_n)$ as follows

Let $i \in \{1, \dots, n-1\}$.

Choose $\underline{b}' \in \mathcal{B}_n$ s.t. $\underline{b}' = \cdots (x_i \cdot x_{i+1}) \cdots$

T_i defn $\longrightarrow$

$$\begin{tikzcd} V_{\underline{b}}^{\otimes n} \arrow[r, "a_{\underline{b}',\underline{b}}"] \arrow[d] & V_{\underline{b}'}^{\otimes n} \arrow[d, "\mathrm{Id}^{\otimes i-1} \otimes c_{V,V} \otimes \mathrm{id}^{\otimes n-i-1}"] \\ V_{\underline{b}}^{\otimes} & V_{\underline{b}'}^{\otimes n} \arrow[l, "a_{\underline{b},\underline{b}'}"] \end{tikzcd}$$

well-defined by Coherence thm. And satisfy braid rel'ns.

$\underline{n=3}$ $\quad \underline{b} = (\cdot\ \cdot)\cdot$

$T_1 = c_{VV} \otimes \mathrm{Id}$

T_2

$$\begin{tikzcd} (V \otimes V) \otimes V \arrow[r, "a"] & V \otimes (V \otimes V) \arrow[d, "\mathrm{Id} \otimes c"] \\ (V \otimes V) \otimes V & V \otimes (V \otimes V) \arrow[l, "a^{-1}"] \end{tikzcd}$$

$T_2 = a^{-1} (\mathrm{Id} \otimes c)\, a$