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Knizhnik-Zamolodchikov equations

Let $\mathfrak{g}$ be a lie algebra $/\mathbb{C}$.

$(\cdot,\cdot) : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{C}$ symmetric bilinear form, nondegenerate and "invariant"

$\hookrightarrow ([x,y],z) = (x,[y,z])$
$\forall x,y,z \in \mathfrak{g}$

Quadratic Lie Alg.

Casimir tensor

Let $\boxed{\Omega \in \mathfrak{g}\otimes\mathfrak{g}}$ be the canonical tensor of $(\cdot,\cdot)$.

That is, if $\{X_a\}$ is a basis of $\mathfrak{g}$ and $\{X^a\}$ is its dual basis,

Let $\Omega = \sum_a X_a \otimes X^a \in \mathfrak{g}\otimes\mathfrak{g}$. This doesn't depend on basis.

eg $\mathfrak{g} = \mathfrak{gl}_m(\mathbb{C})$ (m×m matrices)

$(X,Y) := \mathrm{Tr}(XY)$

$\{e_{ij}\}_{i,j}$ basis, $e_{ij}^* = e_{ji}$

$\rightsquigarrow \Omega = \sum_{i,j} e_{ji}\otimes e_{ij}$

eg Simple Lie algebras

Lemma: If $(\mathfrak{g}, (\cdot,\cdot))$ is a quad. lie alg, and $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir tensor, then

$$[x \otimes 1 + 1 \otimes x, \Omega] = 0 \quad \forall\, x \in \mathfrak{g}.$$

pf $\Omega = \sum_a x_a \otimes x^a$, $\quad x \in \mathfrak{g}$

$$[x, x_a] = \sum_b \underset{\substack{\cap \\ \mathbb{C}}}{\alpha_{ab}} x_b, \qquad [x, x^a] = \sum_b \underset{\substack{\cap \\ \mathbb{C}}}{\beta_{ab}} x^b.$$

Then $\left[x \otimes 1 + 1 \otimes x, \sum_a x_a \otimes x^a\right]$

$$= \sum_a \left([x, x_a] \otimes x^a + x_a \otimes [x, x^a]\right)$$

Coeff of $x_b \otimes x^c$: $\alpha_{cb} + \beta_{bc} = \left([x, x_c], x^b\right) + \left(x_c, [x, x^b]\right)$

$= 0$ by invariance. $\square$

Rk Lemma implies that $\forall\, V_1, V_2$ repns of $\mathfrak{g}$, $\mathfrak{g} \xrightarrow{\pi_j} \mathrm{End}(V_j)$

$$\Omega_{V_1,V_2} : V_1 \otimes V_2 \longrightarrow V_1 \otimes V_2$$

$\quad \shortparallel$

$(\pi_1 \otimes \pi_2)(\Omega)$ is a $\mathfrak{g}$-intertwiner.

Definition Let $\mathfrak{g}$ be a quadratic lie algebra.

Let $n \in \mathbb{Z}_{\geq 2}$, $V_1, \dots, V_n$ repns of $\mathfrak{g}$.

$F := V_1 \otimes \cdots \otimes V_n$

$$\Omega_{ij} \in \mathrm{End}(F) \qquad \underset{(1 \leq i < j \leq n)}{\Omega_{ij}} = \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \overset{i^{th}}{x_a} \otimes \mathrm{Id} \otimes \cdots \otimes \overset{j^{th}}{x^a} \otimes \cdots \otimes \mathrm{Id}$$

$$\boxed{\nabla_{KZ} := d - k \sum_{i<j} \frac{d(z_i - z_j)}{z_i - z_j} \Omega_{ij}}$$

KZ equations.

(k deformation parameter)
$\overset{\in}{\mathbb{C}}$

alternatively

$f(z_1, \dots, z_n) \in F$ or $GL(F)$

$$\boxed{\frac{\partial f}{\partial z_i} = k \sum_{\substack{j \neq i \\ 1 \leq j \leq n}} \frac{\Omega_{ij}}{z_i - z_j} f}$$

Remark $\Omega = \Omega_{21}$

$$\underset{\displaystyle\sum_a x_a \otimes x^a}{\Omega} \;=\; \underset{\displaystyle\sum_a x_a \otimes x^a}{\Omega_{21}}$$

Prop: ① ∇_{KZ} is flat

② If $V_1 = \cdots = V_n$, we have $S_n \circlearrowright F$.

then ∇_{KZ} is S_n-equivariant.

pf (T.S. $[\Omega_{ij}, \Omega_{kl}] = [\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0 \ \forall \ i,j,k,l$ distinct)

① $[\Omega_{ij}, \Omega_{kl}] = 0$ b.c. $[\mathrm{Id}, A] = 0$

$[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = \sum_a [\Omega, x_a \otimes 1 + 1 \otimes x_a] \otimes x^a = 0$

② is obvious: $\sigma \Omega_{ij} \sigma^{-1} = \Omega_{\sigma(i)\sigma(j)}$.

Summary $\forall n \geq 2$ we have a flat connection

$$\begin{array}{c} Y_n(\mathbb{C}) \times (V^{\otimes n}) \\ \downarrow \\ Y_n(\mathbb{C}) := \mathbb{C}^n \setminus \bigcup_{i<j} z_i = z_j \end{array}$$

$\rightsquigarrow$ base $= Y_n(\mathbb{C}) / S_n =: \mathrm{Conf}_n(\mathbb{C})$.

Monodromy representation

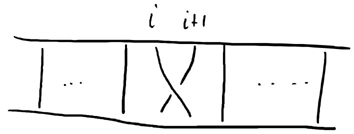
$$\pi_1(\mathrm{Conf}_n(\mathbb{C})) \longrightarrow GL(F)$$

$\| $

B_n Artin's Braid gp.

$\langle T_1, \dots, T_{n-1} \mid T_i T_j = T_j T_i \ \text{ if } \ |i-j| \geq 2, \ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \rangle$

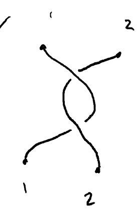
$$\langle T_1, \dots, T_{n-1} \mid T_i T_j = T_j T_i \ \text{ if } \ |i-j| \geq 2, \ \ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \rangle .$$

Pictorially, $T_i =$ 

$\underline{n=2}$

$$\left.\begin{aligned} \frac{\partial f}{\partial z_1} &= \frac{k\,\Omega_{12}\, f}{z_1 - z_2} \\ \frac{\partial f}{\partial z_2} &= \frac{-k\,\Omega_{12}\, f}{z_1 - z_2} \end{aligned}\right\} \quad (\partial_{z_1} + \partial_{z_2}) f = 0$$

$$f = (z_1 - z_2)^{k\Omega} \rightsquigarrow \mu_f\Bigg(\quad \Bigg) = e^{2\pi i k \Omega}$$

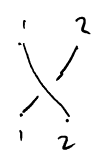


"Half-loop" $\quad V_1 \otimes V_2 \xrightarrow{\hspace{3cm}} V_2 \otimes V_1$

$$(1\,2) \circ \underbrace{e^{\pi i k \Omega}}_{\to\, R_{KZ}}$$

$$\shortparallel$$

$$\mu\Big(\quad \Big)$$



$\underline{n=3}$

$$\nabla = d - k \left(\frac{d(z_1 - z_2)}{z_1 - z_2} \Omega_{12} + \frac{d(z_2 - z_3)}{z_2 - z_3} \Omega_{23} + \frac{d(z_1 - z_3)}{z_1 - z_3} \Omega_{13} \right)$$

$z_1 - z_2 = u \cdot z,$

$z_1 - z_3 = u$ $\Big\}$ $\nabla = \ \ \Big(\frac{du}{u} ($ Central

$$\left.\begin{aligned} z_1 - z_2 &= u \cdot z, \\ z_1 - z_3 &= u, \\ z_2 - z_3 &= u(1-z). \end{aligned}\right\} \quad \nabla = d - k\left(\frac{du}{u} \overbrace{(\Omega_{12} + \Omega_{13} + \Omega_{23})}^{\text{Central}} + \frac{dz}{z}\Omega_{12} + \frac{dz}{z-1}\Omega_{23} \right)$$

$$f = \tilde{f}\, \boxed{u^{k(\Omega_{12} + \Omega_{13} + \Omega_{23})}}$$

and $\tilde{f}$ solves

$$\frac{d\tilde{f}}{dz} = \left(\frac{k\,\Omega_{12}}{z} + \frac{k\,\Omega_{23}}{z-1} \right) \tilde{f} \qquad \left(\begin{array}{l} \text{Drinfeld ODE} \\ \text{w/ } A = k\,\Omega_{12} \\ \quad\; B = k\,\Omega_{23} \end{array} \right)$$

$$\tilde{f}_0 = H_0 \cdot z^{k\Omega_{12}}$$

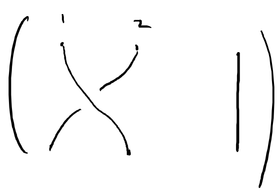
$$\tilde{f}_1 = \cdots$$

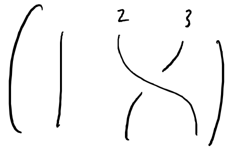
Associator : (Solution near 1)$^{-1}$ (Solution near 0)

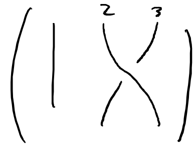
Called KZ associator Φ_{V_1, V_2, V_3}

$\quad = $ Drinfeld Associator w/ $A = k\Omega_{12}$

$\qquad\qquad\qquad\qquad\qquad\quad\;\; B = k\Omega_{23}$

$\mu_{\tilde{f}_0 \boxed{u^{\cdots}}}$  $= (1\,2)\; e^{\pi i k \Omega_{12}}$

$\mu_{\tilde{f}_1 \square}$  $= (1\;2)\; e^{\pi i k \Omega_{23}}$

$\mu_{\tilde{f}_0 \square}$  $= \Phi^{-1}_{V_1, V_2, V_3} \, (2\ 3) \, e^{\pi i k \Omega_{23}} \, \Phi_{V_1, V_2, V_3}$

(Drinfeld) ∇_{KZ} gives a structure of <u>braided tensor category</u> on $\mathrm{Rep}_{fd}(\mathfrak{g})$.

<u>Another example of flat connections</u>

Casimir Connection / Equations

- Let $\mathfrak{g}$ be the simple lie alg assoc. w/ root system R.

$$\nabla_c = d - \sum_{\alpha \in R_+} \frac{d\alpha}{\alpha} K_\alpha$$

$\forall \alpha \in R_+$,

$$K_\alpha := d_\alpha (e_\alpha f_\alpha + f_\alpha e_\alpha)$$

$\mathfrak{g} \ni e_\alpha$ so that

$\mathfrak{g}_{-\alpha} \ni f_\alpha$, $(e_\alpha, f_\alpha) = \frac{1}{d_\alpha}$

$$\begin{array}{c} \mathfrak{h}^{reg} \times V \\ \downarrow \\ \mathfrak{h}^{reg} \end{array}$$

any f.d. repn of $\mathfrak{g}$

Base: $\mathfrak{h}^{reg} = \mathfrak{h} \setminus \bigcup_{\alpha \in R_+} \mathrm{Ker}(\alpha)$

Thm (Millson - Toledano Laredo ; De Concini)

∇_c is flat and W-equivariant.

i.e. $\forall Y \subset R_+$ max'l s.t.

Span(Y) is 2-d,

$$\left[\sum_{\beta \in Y} K_\beta , K_\gamma\right] = 0 \;\; \forall \gamma \in Y$$

Rk W does not act on V.

We have

$$\bar{s}_i = \exp(e_i)\exp(-f_i)\exp(e_i)$$

$$\bar{s}_i^{\,4} = id$$

$\rightsquigarrow$ action of a finite extension $\tilde{W}$ of W on V.

Monodromy Repn

$$\pi_1\left(\mathfrak{h}^{reg}/W\right) \longrightarrow GL(V)$$

$\parallel$

B_W braid gp of W (Brieskorn)